

MAGNETIZED TURBULENT DYNAMO  
IN PROTOGALAXIES

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To my mother

# Abstract

The prevailing theory for the origin of cosmic magnetic fields is that they have been amplified from insignificant seed values to their present values by the turbulent dynamo inductive action driven by the plasma turbulent motions in the protogalactic and galactic medium. Up to now, in calculation of the turbulent dynamo, it has been customary to assume that there is no the back reaction of the magnetic field on the turbulence, as long as the magnetic energy is less than the turbulent kinetic energy. This assumption leads to the kinematic dynamo theory that has been well developed in the past.

However, the applicability of the kinematic dynamo theory to protogalaxies is rather limited. The reason is that in protogalaxies the temperature is very high, and the viscosity is dominated by ions. As the magnetic field strength grows in time because of the dynamo action, the ion cyclotron time becomes shorter than the ion collision time, and the plasma becomes strongly magnetized. As a result, the ion viscosity becomes the Braginskii viscosity, and the magnetic field starts to strongly affect the turbulent motions on the viscous scales. Thus, in protogalaxies the back reaction sets in much earlier, at field strengths much lower than those which correspond to energy equipartition between the field and the turbulence, and the turbulent dynamo becomes what we call the magnetized turbulent dynamo.

The main purpose of this thesis is to lay the theoretical groundwork for the magnetized turbulent dynamo. In particular, we predict that the magnetic energy growth rate in the magnetized dynamo theory is up to ten time larger than that in the kinematic dynamo theory, and this could lead to the dynamo creation of cluster fields. We also briefly discuss how the Braginskii viscosity can aid the development of the inverse cascade of magnetic energy, which happens after the energy equipartition time.

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I would like to thank other people in Princeton, who I worked with: Jeremy Goodman, Gillian Knapp and Scott Tremaine. They were great advisors. I got priceless experience of scientific research, while working with them.

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forms requirements for the PhD defense, and Jane Holmquist for her help in finding journals and books in the university libraries.

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# Chapter 1

## Introduction

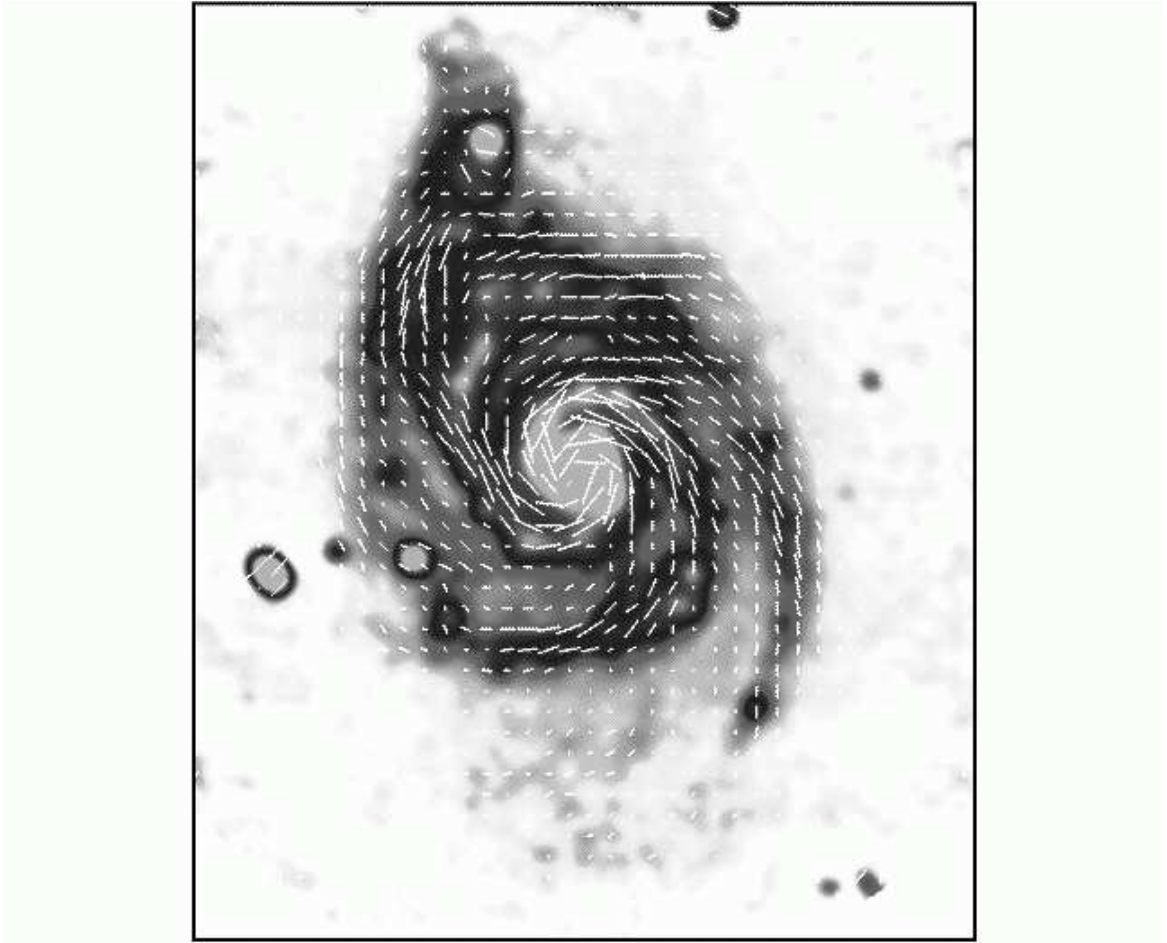
### 1.1 Galactic and Extragalactic Magnetic Fields: Observational Results

One of the most important and challenging questions in astrophysics is the origin of galactic and extragalactic magnetic fields. The existence of interstellar magnetic fields in our Galaxy was first proposed by Alfven in 1943 [1]. In 1949 the polarization of starlight was observed by Hiltner [20] and independently by Hall and Mikesell [17]. The starlight polarization was interpreted as a consequence of light scattering by interstellar dust grains aligned in the galactic magnetic fields. In the same year Fermi employed galactic magnetic field for the acceleration of the cosmic rays and for their confinement in the Galaxy [15]. In 60's the Faraday rotation and the Zeeman effect were measured for different sources distributed over the sky [18, 57]. In 1970 Mathewson and Ford found a large-scale magnetic field in Magellanic Clouds by measuring the polarization of stars in the Magellanic System [33]. In 1974 Manchester first correctly measured the galactic magnetic field in the vicinity of the Sun (within

$\approx 2$  Kpc) by measuring the Faraday rotation for thirty eight nearby pulsars [32]. He found that the local field has a longitudinal component in the Galactic plane. In 1989 Rand and Kulkarni confirmed and significantly improved the measurements of the local magnetic field [42]. They analyzed the Faraday rotation measures for nearly two hundred pulsars within  $\approx 3$  Kpc, and they found that the uniform component of the local magnetic field has a strength of  $\approx 1.6 \mu\text{G}$  towards a galactic longitude of  $96^\circ$ , with a reversal of the field at a distance about 0.6 Kpc towards the galactic center. The random field component was estimated by Rand and Kulkarni as  $\sim 5 \mu\text{G}$ . Since early 90's a significant progress has been made in measuring magnetic fields in our Galaxy and in other spiral galaxies, by measuring polarization of starlight, by measuring the Faraday rotation from pulsars and extragalactic radio sources, by detecting synchrotron radiation from relativistic electrons, and by measuring the Zeeman splitting of spectral lines; see recent excellent review papers [4, 24, 58]. The observations indicate that galaxies possess magnetic fields with strengths of several microgauss (up to several tens of microgauss). These fields have uniform components, with strengths comparable to those of the random components (e. g. in the vicinity of the Sun the uniform/random field strength ratio is about 1/2). The uniform field components lie in the galactic disks and have typical correlation lengths from several hundreds parsecs to one kiloparsec. Figure 1.1 shows the magnetic field pattern for the galaxy M51. However, most galaxies do not show such a nice regular field pattern as M51 does. As a result, the question about the prevailing geometrical structure of magnetic fields in spiral galaxies is still open, see Figure 1.2. The magnetic field in Milky Way exhibits reversals with radius, which suggests a bisymmetric spiral structure for it, see Figure 1.2(B).

It is very important to know that magnetic fields may exist in very young galaxies

## M51 6cm Total Intensity + Magnetic Field



**Copyright: MPIfR Bonn (R. Beck, C. Horellou & N. Neininger)**

Figure 1.1: Magnetic field vectors (white bars) for the galaxy M51 (measured by fitting a model of the polarized synchrotron emission to radio observations of M51). The map is superimposed on a M51 image in 6 cm. The magnetic field lines approximately follow the spiral structure. By courtesy of Beck, Horellou, Neininger, and the MPIfR collaboration [34].

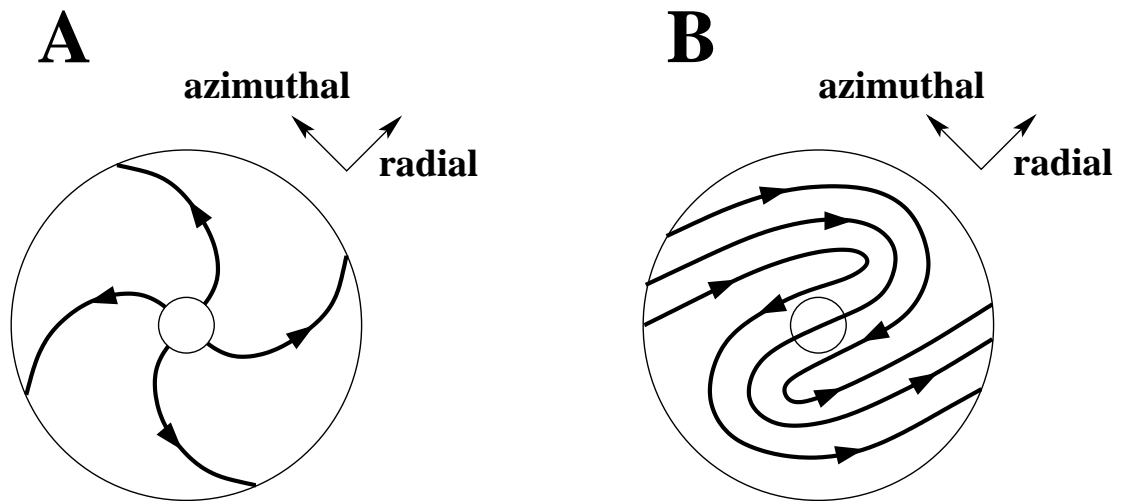


Figure 1.2: Two possible magnetic field configurations in disk galaxies, face-on view. The bold lines are magnetic field lines with the field direction shown by the arrows. A: axisymmetric spiral structure, no radial reversals of the field (this structure is predicted by the galactic dynamo theory); B: bisymmetric spiral structure, there are radial reversals of the field (this structure arises in the rotating galactic disk if the field has primordial origin [58]).

(at high redshifts) [58]. In 1992 Wolfe, Lanzetta and Oren found that the probability for Faraday rotation for radio QSOs (quasi-stellar objects) is significantly higher in damped Ly $\alpha$  systems [56]. These systems are associated with early forming galactic disks. Wolfe *et al.* estimated the magnetic field in two damped Ly $\alpha$  systems with  $z \approx 2$  as a few microgauss. More recent extensive observations have confirmed this result. Thus, magnetic fields with similar spatial scale and strength to those in the local universe exist at redshifts above  $z = 1$ , see review papers [24, 38, 58]. The field strength estimates all fall in the range of 1 to  $5 \mu\text{G}$ , and there is an absence of observational evidence for cosmological evolution of magnetic field strengths! In addition, there also exist some recent observational data which indirectly indicate the existence of rather strong magnetic fields in the distant past. The observations of  $^6\text{Li}$  and  $^7\text{Li}$  abundances in old metal-pure halo stars by Lemoine *et al.* indicate a massive production of lithium isotopes by cosmic rays in a very early phase of the Milky Way Galaxy [30]. Without magnetic fields the cosmic rays would escape and would not be able to produce the lithium isotopes. Other indirect evidence for magnetic fields in the past is the primordial star formation process, which must have happened according to the observed metallicities of globular clusters. To form stars at early times ( $z \sim 3$ ), primordial magnetic fields are believed to be required in order to remove significant angular momentum of self-gravitating gas by the magnetic braking effect [40].

Strong magnetic fields have been observed not only in galaxies, local and at high redshifts, but also in clusters of galaxies [14, 24, 58]. The most recent direct observations of the Coma Cluster by Fusco-Femiano *et al.* [16] manifest a volume averaged intracluster magnetic field of  $\sim 0.15 \mu\text{G}$  in the cluster. Fusco-Femiano *et al.* used the inverse Compton scattering of relativistic electrons on the cosmic microwave background photons to model the hard nonthermal X-ray radiation flux from the Coma

Cluster. As a result, they estimated the electron density in the cluster. Then, they used the synchrotron radio flux to estimate the field strength. In 2000 Sarazin and Kempner revised the results of Fusco-Femiano *et al.* . Sarazin and Kempner used non-thermal bremsstrahlung radiation models for the hard X-ray emission from the Coma Cluster, and depending on the model they used, they estimated the field strength to be ranging from  $\sim 0.4 \mu\text{G}$  (as also implied by equipartition in the radio halo) to  $\sim 6 \mu\text{G}$  (as the typical field in individual galaxies in the cluster) [44].

Thus, galaxies, early galaxies (at  $z \sim 2$  redshift) and galaxy clusters do have strong magnetic fields, with the strengths  $0.1 \mu\text{G} \lesssim B \lesssim 10 \mu\text{G}$ ! Where did these strong fields come from?

## 1.2 The Origin of Galactic and Extragalactic Magnetic Fields: Primordial and Galactic Dynamos

The prevailing theory for the origin of strong cosmic magnetic fields is that they were produced by the turbulent dynamo inductive action driven by the fluid motions in galactic and/or protogalactic medium. The turbulent dynamo works as follows. The astrophysical plasmas are very hot and they have low densities (e. g. see Table 1.1). Therefore, the effect of electrical resistivity is negligible over an extremely broad range of scales. As a result, the magnetic field lines are frozen into plasma [29, 55]. Because in a turbulent plasma the distance between any two infinitesimally neighboring points increases exponentially in time (the Kolmogorov-Lyapunov exponentiation), the magnetic field lines are stretched and the field strength grows exponentially fast [22, 55], see the left plot in Figure 1.3. There is an additional mechanism for the growth of

the mean field strength in the case of three-dimensional turbulent dynamos, which is called the Zeldovich “figure 8” mechanism [29, 55]. This mechanism is shown in the right plot of Figure 1.3, where the closed magnetic field lines are frozen into the torus. The field strength is increased exponentially fast in time by repetition of first, stretching the torus, second, twisting it into “figure 8”, and third, folding it.

Yet full understanding of all stages of production of the strong galactic and extragalactic magnetic fields by the turbulent dynamo action has not been achieved. There are two alternative theories on how and when the fields have been produced.

The first theory, the galactic dynamo theory, also known as the  $\alpha$ - $\Omega$  dynamo theory, states that the fields have been primarily amplified in differentially rotating galactic disks after the galaxies had been formed. This theory considers the evolution of a large-scale mean magnetic field in a galactic disk. The turbulent dynamo action and the differential rotation of the disk are parametrized by transport coefficients. These transport coefficients enter the evolution equation for the mean field and they represent the destruction and the induction of the mean field [4, 35, 36, 37, 50, 53, 54, 55]. The typical time scale of the mean field growth in the galactic disk is approximately equal to the disk rotation period,  $\sim 300$  million years. The galactic dynamo theory is a beautiful theory, but it involves several crucial unsolved problems [10, 19, 25, 26, 43, 58, 52]. First, in the  $\alpha$ - $\Omega$  theory it seems to be extremely difficult to expel a fraction of the magnetic flux from the galactic disk in order to produce the net magnetic flux [25, 41]. The second problem is that the effect of small-scale fields on the amplification of the mean magnetic field remains unclear. These small-scale fields are amplified on the smallest turbulent eddy turnover time scale, much faster than the mean field is being amplified (see Figure 1.5). Thus, the small-scale fields must saturate and possibly significantly alter the  $\alpha$ - $\Omega$  dynamo ac-

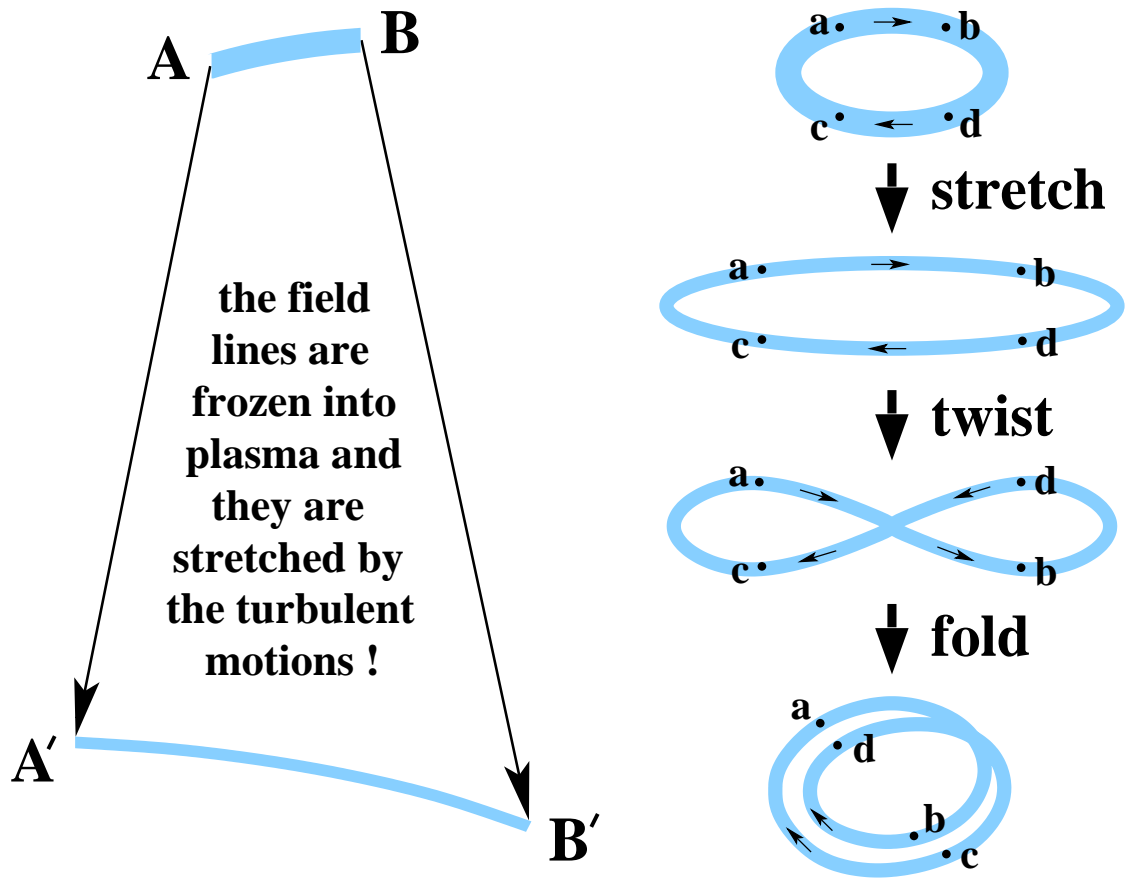


Figure 1.3: Left plot: stretching of magnetic field lines in a turbulent plasma because of the Kolmogorov-Lyapunov exponential divergence of closely neighboring points. Right plot: three-dimensional Zeldovich “figure 8” dynamo mechanism.

tion well before the mean field is amplified appreciably [25, 27]. The third problem is that the galactic dynamo theory requires rather large seed fields in the distant past, in order to successfully build up the fields observed at the present time. The Biermann battery itself and other similar effects do not seem to be able to provide the required seed fields [5, 25]. Finally, as we wrote in Section 1.1, the observational data indicate that galaxy clusters and galaxies at low and high redshifts possess magnetic fields of up to several microgauss. It is hard to explain these strong fields by the galactic dynamo theory [58]. On one hand, the galactic dynamo just did not have enough time to build microgauss fields in early galaxies at, say, redshift  $z \sim 2$ . On the other hand, the fields in clusters of galaxies are several orders of magnitude larger than the fields that would be obtained from galactic ejecta alone. Thus, sufficiently strong initial magnetic fields must be generated during the pregalactic era!

In this thesis we accept the second theory for the origin of cosmic magnetic fields, *the primordial dynamo theory*, which states that the galactic and extragalactic magnetic fields have primarily been produced in protogalaxies, i. e. before the galaxies were formed [25, 26, 27, 40]. Of course, these fields were subsequently modified in the rotating galactic disks after the galaxies were formed [5].

In order to understand how the magnetic fields can be built up in protogalaxies, let us first discuss the physical conditions that were present there. The typical values of physical parameters in a protogalaxy are given in Table 1.1, and the hierarchy of scales is displayed on the  $x$ -axis of Figure 1.4. The most important facts are the following.

- First, the gas is very hot in a protogalaxy, so it is fully ionized. Therefore, the viscosity is dominated by ions, not by neutrals. In addition, the Spitzer resistivity [49] is tiny, and thus, the resistive cutoff scale for the magnetic field,

| Parameter                                    | Notation   | Value*                               | Scaling*                                   |
|--|--|--------------------------------------|--|
| Physical Quantities                          |  |                                      |  |
| <b>total mass, g</b>                         | <b><math>M</math></b>                            | <b><math>2 \times 10^{45}</math></b> | <b><math>\sim 10^{12} M_{\odot}</math></b> |
| <b>total/baryon mass ratio</b>               | <b><math>\xi</math></b>                          | <b>10</b>                            |  |
| temperature, K                               | $T \sim T_i \sim T_e$                            | $2 \times 10^6$                      | $ML^{-1}$                                  |
| ion & $e^-$ density, $\text{cm}^{-3}$        | $n$  | $5 \times 10^{-4}$                   | $\xi^{-1} ML^{-3}$                         |
| neutral density, $\text{cm}^{-3}$            |  | 0                                    |  |
| ion thermal speed, $\text{cm/s}$             | $V_T$  | $2 \times 10^7$                      | $M^{1/2} L^{-1/2}$                         |
| ion viscosity, $\text{cm}^2/\text{s}$        | $\nu \equiv \nu_i \sim V_T^2 t_i$                | $5 \times 10^{26}$                   | $\xi M^{3/2} L^{1/2}$                      |
| $e^-$ thermal speed, $\text{cm/s}$           | $(m_i/m_e)^{1/2} V_T$                            | $9 \times 10^8$                      | $M^{1/2} L^{-1/2}$                         |
| $e^-$ viscosity, $\text{cm}^2/\text{s}$      | $\sim (m_e/m_i)^{1/2} \nu_i$                     | $10^{25}$                            | $\xi M^{3/2} L^{1/2}$                      |
| neutral viscosity, $\text{cm}^2/\text{s}$    |  | 0                                    |  |
| magnetic diffusivity, $\text{cm}^2/\text{s}$ | $\eta_s$   | $8 \times 10^4$                      | $M^{-3/2} L^{3/2}$                         |
| smallest eddy speed, $\text{cm/s}$           | $V_{\nu} \sim R^{-1/4} V_T$                      | $2 \times 10^6$                      | $\xi^{1/4} M^{3/4} L^{-1/2}$               |
| Dimensionless numbers                        |  |                                      |  |
| hydrodynamic Reynolds                        | $R \sim V_T L / \nu$                             | $3 \times 10^4$                      | $\xi^{-1} M^{-1}$                          |
| magnetic Reynolds                            | $R_m \sim V_T L / \eta_s$                        | $2 \times 10^{26}$                   | $M^2 L^{-1}$                               |
| Prandtl number                               | $Pr \sim \nu / \eta_s$                           | $6 \times 10^{21}$                   | $\xi M^3 L^{-1}$                           |
| field/smallest eddy energy                   | $B^2 / 4\pi m_p n V_{\nu}^2$                     | $3 \times 10^{13} B^2$               | $\xi^{1/2} M^{-5/2} L^4 B^2$               |
| Length Scales                                |  |                                      |  |
| <b>system size, cm</b>                       | <b><math>L</math></b>                            | <b><math>6 \times 10^{23}</math></b> | <b><math>\sim 0.2 \text{ Mpc}</math></b>   |
| viscous cutoff scale, cm                     | $2\pi k_{\nu}^{-1} \sim R^{-3/4} L$              | $3 \times 10^{20}$                   | $\xi^{3/4} M^{3/4} L$                      |
| ion mean free path, cm                       | $\lambda_i = V_T t_i \sim R^{-1} L$              | $7 \times 10^{19}$                   | $\xi M L$                                  |
| ion gyroradius, cm                           | $r_i = V_T / \omega_i$                           | $2 \times 10^3 / B$                  | $M^{1/2} L^{-1/2} B^{-1}$                  |
| $e^-$ mean free path, cm                     | $\sim (m_e/m_i)^{1/2} \lambda_i$                 | $2 \times 10^{18}$                   | $\xi M L$                                  |
| $e^-$ gyroradius, cm                         | $\sim (m_e/m_i)^{1/2} r_i$                       | $50 / B$                             | $M^{1/2} L^{-1/2} B^{-1}$                  |
| resistive cutoff scale, cm                   | $2\pi k_{\eta_s}^{-1} \sim Pr^{-1/2} R^{-3/4} L$ | $4 \times 10^9$                      | $\xi^{1/4} M^{-3/4} L^{3/2}$               |
| Debye length, cm                             | $(k_B T / 2\pi n e^2)^{1/2}$                     | $6 \times 10^5$                      | $\xi^{1/2} L$                              |
| Time Scales                                  |  |                                      |  |
| gravitational collapse time, s               | $\sim L / V_T$                                   | $3 \times 10^{16}$                   | $M^{-1/2} L^{3/2}$                         |
| largest eddy turnover time, s                | $\sim L / V_T$                                   | $3 \times 10^{16}$                   | $M^{-1/2} L^{3/2}$                         |
| smallest eddy turnover time, s               | $\sim R^{-1/2} L / V_T$                          | $2 \times 10^{14}$                   | $\xi^{1/2} L^{3/2}$                        |
| ion collision time, s                        | $t_i$  | $3 \times 10^{12}$                   | $\xi M^{1/2} L^{3/2}$                      |
| ion cyclotron frequency, $\text{s}^{-1}$     | $\omega_i = eB / m_i c$                          | $9 \times 10^3 B$                    | $B$  |
| $e^-$ collision time, s                      | $\sim (m_e/m_i)^{1/2} t_i$                       | $7 \times 10^{10}$                   | $\xi M^{1/2} L^{3/2}$                      |
| $e^-$ cyclotron frequency, $\text{s}^{-1}$   | $(m_i/m_e) \omega_i$                             | $2 \times 10^7 B$                    | $B$  |

Table 1.1: Physical parameters in a protogalaxy calculated for a fully ionized hydrogen plasma, assuming definite values for the parameters printed in boldface. \* The field  $B$  in column “Value” is expressed in gauss, the Coulomb logarithm is taken to be 30.

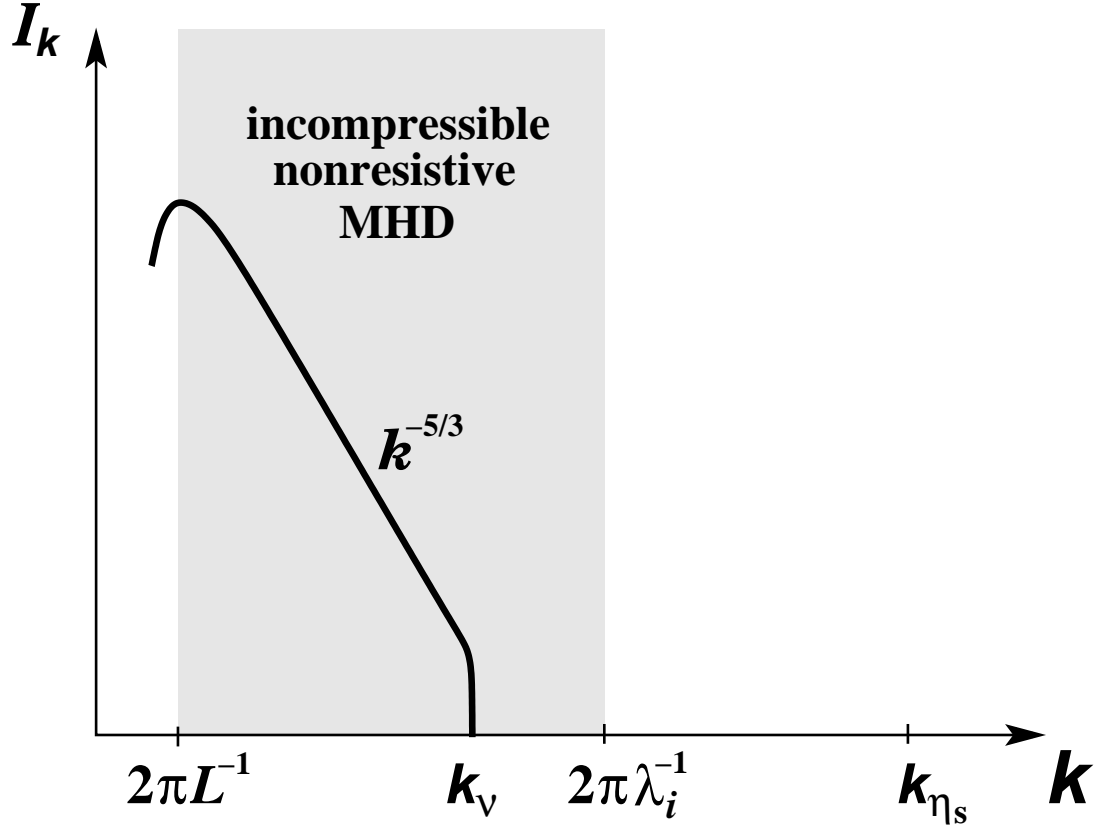


Figure 1.4: The Kolmogorov spectrum of kinetic energy,  $I_k$ , is shown by the thick solid line. The  $x$ -axis has the wave number  $k$  plotted on it, and demonstrates the hierarchy of scales in a protogalaxy:  $L$  is the system size,  $2\pi k_\nu^{-1}$  is the viscous cut-off scale,  $\lambda_i$  is the ion mean free path, and  $2\pi k_{\eta_s}^{-1}$  is the resistive scale. Note that  $L \gg 2\pi k_\nu^{-1} \gg \lambda_i \gg 2\pi k_{\eta_s}^{-1}$ . The equations for a nonresistive incompressible MHD turbulence are valid only inside the range of scales shown by the shaded area (see also footnote 1 on page 12).

$2\pi k_{\eta_s}^{-1}$ , is extremely small compared to the viscous cutoff scale for the turbulent velocities,  $2\pi k_\nu^{-1}$ . As a result, we can neglect resistivity as long as we consider scales  $\gg 2\pi k_{\eta_s}^{-1}$ , see Figure 1.4.

- Second, the Reynolds number is very large. Therefore, the turbulence develops over a very broad range of scales, starting at the largest scale,  $L$ , and ending at the viscous cutoff scale,  $2\pi k_\nu^{-1}$  (see Figure 1.4). The velocities at the largest scale are of the order of the sound speed, which is approximately equal to the thermal speed. All turbulent velocities at smaller scales are smaller. As a result, we can treat the plasma as incompressible on scales  $2\pi k^{-1} \ll L$ .
- Third, the ion mean free path  $\lambda_i$  is considerably shorter than the viscous cutoff scale for the turbulent velocities,  $2\pi k_\nu^{-1}$ . The Debye length and the resistivity scale are even much shorter than  $\lambda_i$ . Thus, we can use the single-fluid magneto-hydrodynamic (MHD) equations for description of a nonresistive incompressible plasma on scales  $\lambda_i \lesssim 2\pi k^{-1} \lesssim L$ .<sup>1</sup>
- Finally, it is very important that the cyclotron period of ions in the magnetic field is shorter than the ion collision time,  $\omega_i^{-1} \ll t_i$ , provided that the field strength is larger than  $\sim 10^{-16}$  G (see Table 1.1). On the other hand, the energy of the magnetic field becomes comparable to the kinetic energy of the smallest turbulent eddies (which are on the viscous cutoff scale) if the field exceeds  $\sim 10^{-7}$  G. As a result, there is a very broad range of magnetic field strength values at which the magnetic pressure and tension are still negligible, while the presence of the field is already important. This is because the plasma

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<sup>1</sup> We should consider scales larger than  $\lambda_i$  in order to use MHD description for the plasma. However, for estimation purposes, one can use MHD equations even for smaller scales,  $2\pi k^{-1} \ll \lambda_i$ , if one reduces the molecular viscosity by the ratio of the scale to the ion mean free path, i. e. by factor  $1/(k\lambda_i) \ll 1$ .

is strongly magnetized, and the magnetic field controls the microscopic motions of ions, so that the plasma viscosity is different from that in a field-free plasma.

The primordial dynamo theory is as follows. We believe that there are five major stages of the production of the strong magnetic fields. Four of them happen in protogalaxies, and one happens in galaxies, see Figure 1.5. These stages are the following:

1. During the first stage, in a protogalaxy undergoing gravitational collapse, the Biermann battery [6] builds a seed magnetic field linearly in time. The Biermann battery requires the pressure to be non-barotropic, which is achieved behind hydrodynamic shocks driven by the gravitational instability [28]. The resulting seed field is of the order of  $10^{-21}$ – $10^{-19}$  G at the largest scales [13, 28, 40]. The Biermann battery inductive action is proportional to the vorticity of the turbulence, which is  $R^{1/2}$  times larger at the viscous scales compared to the vorticity at the largest scales ( $R \gg 1$  is the Reynolds number). Therefore, the seed field at the smallest scales of the turbulence (the viscosity scales), is about  $\sim 10^{-18}$  G. The time scale of the Biermann battery action is approximately equal to the free-fall time in a protogalaxy,  $\sim 1$  billion years.
2. During the second stage, when the plasma is unmagnetized and  $\omega_i^{-1} \gg t_i$ , the seed field is exponentially amplified by *the kinematic turbulent dynamo* inductive action. The kinematic dynamo builds the field up to approximately  $10^{-16}$  G, when the plasma becomes magnetized,  $\omega_i^{-1} \ll t_i$ , and the field becomes strong enough to affect microscopic processes in plasma, such as the viscosity. The time scale of the kinematic dynamo is very short, it is of the order of the smallest eddy turnover time,  $\sim 5$  million years. The main assumption of the kinematic dynamo theory and, basically, the strict definition of it, are that the growing

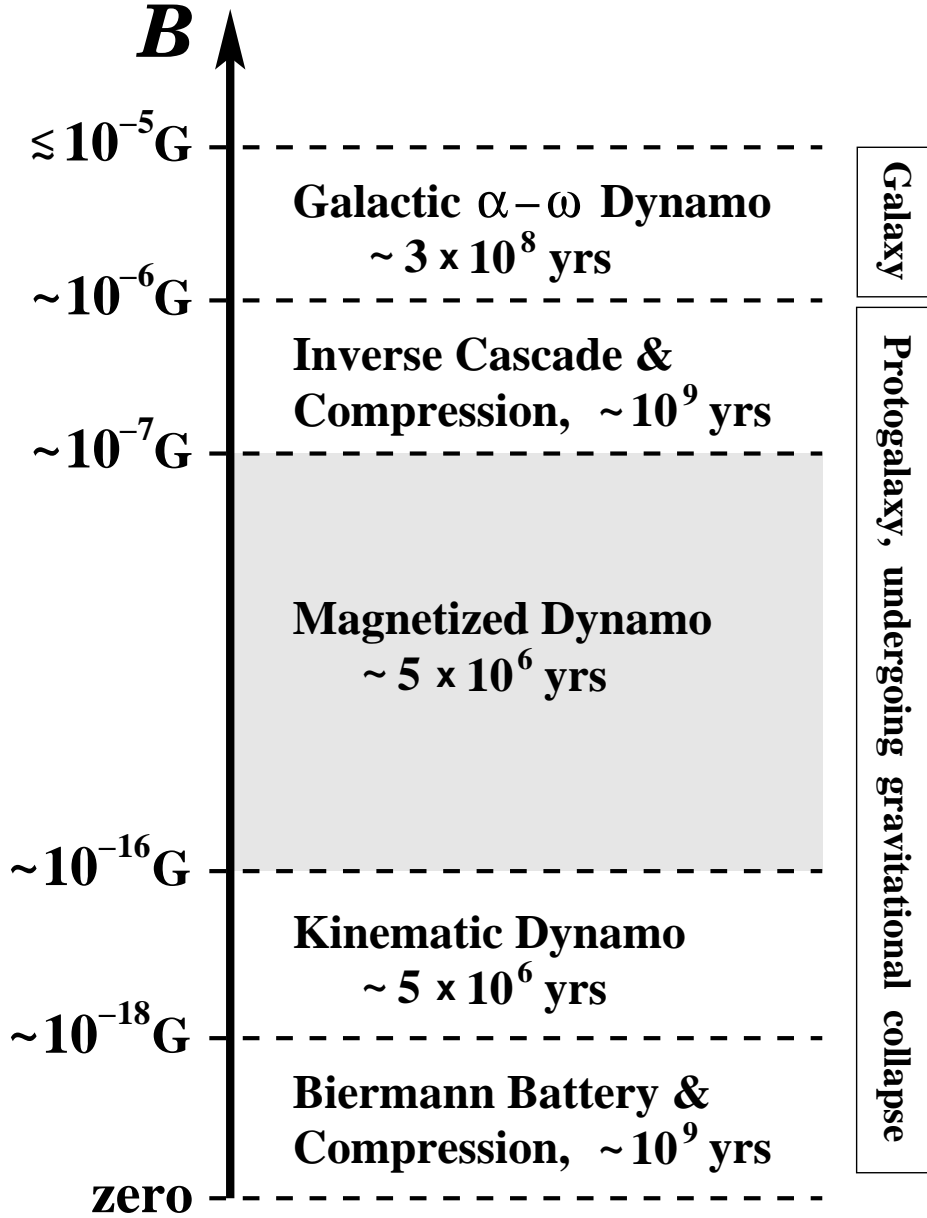


Figure 1.5: There are five major stages of the production of strong galactic and extragalactic magnetic fields, as the field strength grows from zero up to  $\lesssim 10^{-5}$  gauss. The results of this thesis apply to the magnetized turbulent dynamo stage in protogalaxies, shaded on the plot. This stage makes the largest contribution to the built up of the magnetic fields.

magnetic field stays so weak, that it does not affect the fluid motions, i. e. that there is no *the back reaction* of the field on the turbulence. The kinematic dynamo has been intensively studied during the last several decades (see the references given in Section 2.1). Up to now it has been customary to assume that there is no the back reaction, and the main assumption of the kinematic dynamo is valid, as long as the magnetic energy is less than the turbulent kinetic energy, so that the Lorentz forces are small. However, the plasma becomes strongly magnetized and the back reaction sets in much earlier, at field strengths much lower than those which correspond to the energy equipartition between the field and the turbulence (see Figure 1.5). Thus, the applicability of the kinematic dynamo theory to protogalaxies is rather limited. In Section 2.1 we discuss the main results obtained for the kinematic dynamo theory. The results we obtain in this thesis reduce to those of the kinematic dynamo theory in the limit of very weak field strength (i. e. when the plasma is unmagnetized, and there is no the back reaction).

3. The third stage starts when the field grows above  $\sim 10^{-16}$  G, the ion cyclotron time in the magnetic field becomes shorter than the ion collision time, and the plasma becomes strongly magnetized. As a result, the magnetic field starts to strongly affect the dynamics of the turbulent motions on the viscous scales <sup>2</sup>, by completely changing the viscosity (see Section 2.2), despite the fact that the field energy is still very small compared to the kinetic energy of the turbulence! We call this stage as *the magnetized turbulent dynamo* (see Figure 1.5). In previous theories this stage has not been recognized. The main goal of this thesis is to

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<sup>2</sup> Note that the turbulent motions on the viscous scales are the most important in the dynamo theory. This is because the magnetic field is primarily amplified by the turbulent eddies on these scales. These eddies are the smallest ones, they have the shortest turnover times and produce the largest velocity shearings [27].

construct a theoretical model for it. The time scale of the magnetized dynamo is roughly about the same as the time scale of the kinematic dynamo,  $\sim 5$  million years.

4. So far the field scale is of the order of the viscous scale or less, and the magnetic field is incoherent in space. The fourth stage starts when the magnetic field strength grows up to  $\sim 10^{-7}$  G. The field energy becomes comparable to the kinetic energy of the smallest turbulent eddies (the energy equipartition), and the Lorentz forces become dynamically important in the plasma. During this stage the turbulent motions are dissipated by the growing field, the turbulent energy spectrum becomes truncated at larger and larger scales, and the turbulent energy is eventually transferred into a large-scale strong magnetic field with its energy comparable to the kinetic energy of the fluid motions on the largest scales in the protogalaxy (which is approximately the same as the thermal energy). This process is called the inverse cascade [8, 4, 27, 39, 51]. We discuss it only qualitatively in Chapter 5. The time scale available for the inverse cascade process is of the order of the largest eddy turnover time,  $\sim 1$  billion years. The inverse cascade may not have time to amplify the field up to microgauss values, which are observed in galaxies. A crucial question is how far it goes. This thesis addresses this question and is concerned with the rate of the magnetic field built up by the magnetized dynamo.
5. Finally, the fifth stage is the galactic dynamo, which happens in the differentially rotating galactic disc after the galaxy is formed (see the discussion of the galactic dynamo above and Figure 1.5). This process modifies the strong field that was built up in the protogalaxy in advance [21]. The galactic dynamo may also further amplify the mean magnetic field in the disk by a factor of three or so,

on a time scale of the order of the rotation time in the galaxy,  $\sim 300$  million years. The galactic dynamo theory is beyond the scope of this thesis.

As we discussed in the beginning of this section, the galactic mean field dynamo theory seems to be unable, by itself, to build magnetic fields from the seed fields of  $\sim 10^{-18}$  gauss, which are provided by the Biermann battery action, up to  $\gtrsim 10^{-6}$  gauss, which exist in the present Universe. The observational evidences and the theoretical problems of the galactic dynamo theory, discussed above, persuade us to accept the primordial dynamo theory, which states that the observed cosmic fields were primarily built up in protogalaxies. The primordial dynamo theory seems to be free of the shortcomings of the galactic dynamo theory. The turbulent dynamo in protogalaxies does seem to be able to amplify the seed field up to  $\sim 10^{-7}$  G in a time interval  $\lesssim 10^7$  years, which is negligible in comparison to the Hubble time. However, up to now, there was one major deficiency in the primordial dynamo theory. It is well known that the turbulent dynamo builds the magnetic fields first on small subviscous scales [27], while the observed cosmic fields have rather large correlation lengths. Therefore, the magnetic field lines must be unwrapped on the small scales by the Lorentz tension forces, while the field energy is transferred and amplified on larger scales during the inverse cascade stage. Up to now, there was a strong argument against the possibility of such unwrapping. This argument is as the following. Both theoretical calculations and numerical simulations [11, 46, 48] show that the magnetic field, built up by the standard dynamo action, has folding structure with the wave number perpendicular to the field lines much larger than the wave number parallel to the field lines, as demonstrated in Figure 2.1. Therefore, the Lorentz tension forces, which are proportional to the parallel wave number, are not very large. As a result, the standard isotropic viscous forces, calculated for an unmagnetized or partially

ionized plasma, can balance the Lorentz tension forces <sup>3</sup>, and block the unwrapping of the field lines (see Chapter 5). At the same time, the perpendicular wave number can grow in time, and the perpendicular scale of the field lines can unrestrictively decrease <sup>4</sup>.

This anti-unwrapping argument [11] applies only if the viscous forces are isotropic. However, in the case of the magnetized turbulent dynamo in a fully ionized plasma, the magnetic field controls the microscopic motions of particles on the viscous scales. The transport of ion momentum across the field lines is inhibited, and the viscosity stress is given by the Braginskii tensor [see equation (2.28) and the discussion before it]. As a result, in the magnetized dynamo theory the viscous forces do not prevent the unwrapping of the magnetic field lines by the Lorentz tension forces on the small scales and do allow the field to become converted into a large-scale field during the inverse cascade stage in a protogalaxy (see Chapter 5 for more discussion). On the other hand, the theory of turbulent dynamos with the Braginskii viscosity has not been worked out. It turns out that the dynamo theory is qualitatively modified by including the Braginskii viscosity. The main purpose of this thesis is to lay the theoretical groundwork for the magnetized turbulent dynamos.

### 1.3 The Thesis Preview

Let us briefly outline the thesis content.

In Chapter 2 we give the main equations and the main results of the kinematic dynamo theory (in Section 2.1), and we formulate the basic equations of the magnetized dynamo theory (in Section 2.2).

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<sup>3</sup> The magnetic pressure is balanced by the hydrodynamic pressure in an incompressible plasma.

<sup>4</sup> Until it reaches the scales at which resistivity becomes important, see Figure 1.4 on page 11.

In Chapter 3 we calculate the statistics of turbulent velocities,  $\mathbf{V}$ , in strongly magnetized plasmas. In Section 3.1 we write down the quasilinear expansion in time of the MHD equations for both the velocities and the magnetic field, similar to the expansion used by Kulsrud and Anderson [2, 27]. In section 3.2 we make use of the Laplace transformation in time to find the turbulent velocities. We assume that the tensor  $b_{\alpha\beta}$ , which is the product of two unit vectors in the direction of the magnetic field, see equation (2.38), can be taken to be constant in space in the beginning of the expansion in time,  $b_{\alpha\beta} = \hat{b}_\alpha \hat{b}_\beta = \text{const}$  at zero time. This is our first hypothesis. It basically relies on our assumption that in the case of the magnetized turbulent dynamo the magnetic field has a folding structure similar to the one that exists in the case of the kinematic turbulent dynamo (see Figure 2.1 and the discussion in Section 3.2). We find that there are velocity modes which are not damped by the Braginskii viscous forces. These undamped velocity modes grow unrestrictively in time unless we incorporate the non-linear inertial terms of the MHD equations into our quasilinear expansion. In Section 3.3 we argue that the non-linear damping of the growing velocity modes may be included into our theory by allowing for rotation of velocity vectors relative to the magnetic field unit vectors <sup>5</sup>. This is our second hypothesis. In section 3.4 we again solve for the turbulent velocities by now including our effective rotational damping into the MHD equations and by making use of the Fourier transformation in time. As a result, we obtain the correlation functions for the turbulent velocities in a strongly magnetized plasma, similar to those given by equations (2.4) and (2.5) for the Kolmogorov velocities in the kinematic dynamo case. We find that, contrary to the Kolmogorov velocities, the turbulent velocities in the magnetized plasma are strongly anisotropic, as one might expect, because the

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<sup>5</sup> This rotation is essentially due to Coriolis forces that make the velocity rotate differently than the magnetic field direction.

magnetic field sets “a preferred axis in space”.

In Chapter 4 we use the correlation functions for the turbulent velocities found in Chapter 3 to calculate the energy spectrum of random magnetic fields in the magnetized plasma. We start with calculations of the total magnetic energy growth rate in Section 4.1. In Section 4.2 we derive the mode coupling integro-differential equation for the evolution of the magnetic energy spectrum. Our mode coupling equation is a more general version of equation (2.21) of Kulsrud and Anderson obtained in the kinematic dynamo theory [27]. In Section 4.3 we consider the magnetic energy spectrum on small subviscous scales. On these scales our mode coupling equation greatly simplifies and becomes a homogeneous differential equation, similar to the equation (2.25) of Kulsrud and Anderson. We find the Green’s function solution of our differential equation.

Finally, in Chapter 5 we give our conclusions. We discuss the possibilities of further research on the magnetized turbulent dynamos. We also discuss the peculiarities of the inverse cascade in a strongly magnetized turbulent plasma.

# Chapter 2

## Basic Dynamo Equations

### 2.1 The Kinematic Turbulent Dynamo

The main assumption of the kinematic turbulent dynamo theory is that the magnetic field is weak and it does not affect the turbulent motions in plasma. Thus, the magnetic field  $\mathbf{B}$ , frozen into plasma, is evolved as a passive vector by the turbulent velocities  $\mathbf{U}$  [29]:

$$\partial_t B_\alpha = U_{\alpha,\beta} B_\beta - U_\beta B_{\alpha,\beta}. \quad (2.1)$$

Here and below we always assume summation over repeated indices. In order to shorten notations, we frequently use  $\partial_t \stackrel{\text{def}}{=} \partial/\partial t$ , and spatial derivatives are assumed to be taken with respect to all indices that are listed after “,” signs <sup>1</sup>. We also assume that the turbulence is incompressible (see Section 1.2). In the kinematic dynamo theory the turbulent velocities  $\mathbf{U}$  are assumed to be hydrodynamic and to be independent of the magnetic field.

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<sup>1</sup> For example,  $\partial_t B_\alpha \equiv \partial B_\alpha / \partial t$ ,  $(U_\alpha B_\beta)_{,\gamma} \equiv B_\beta (\partial U_\alpha / \partial x_\gamma) + U_\alpha (\partial B_\beta / \partial x_\gamma)$ , and  $U_{\alpha,\beta\gamma} \equiv \partial^2 U_\alpha / \partial x_\beta \partial x_\gamma$ .

A vast number of papers have been written about the kinematic turbulent dynamo in order to answer three main questions: first, “How fast is the total magnetic energy built up by the kinematic dynamo action”, second, “What is the magnetic energy spectrum on different scales?”, and third, “What is the resulting geometrical structure of the magnetic field?” [9, 11, 22, 23, 27, 31, 35, 36, 39, 45, 46, 47, 50, 51, 55]. It turns out that in most cases the evolution of magnetic field is entirely determined by the specified two-point statistics of the turbulent velocities.

Following the assumptions usually made in the kinematic dynamo theory, we assume that the turbulence is incompressible, homogeneous, isotropic and stationary (as it actually is in protogalaxies). We also neglect the helical part of the turbulence<sup>2</sup>. We assume that the statistics of the Fourier coefficients of the turbulent velocities,

$$\tilde{U}_{\mathbf{k}\alpha}(t) = \frac{1}{L^3} \int_{-L/2}^{L/2} U_{\alpha}(t, \mathbf{r}) e^{-i\mathbf{k}\mathbf{r}} d^3\mathbf{r}, \quad (2.2)$$

$$\tilde{U}_{\mathbf{k}\alpha}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{U}_{\mathbf{k}\alpha}(t) e^{i\omega t} dt, \quad (2.3)$$

is given by the following formulas [27]:

$$\langle \tilde{U}_{\mathbf{k}\alpha}(\omega) \rangle = 0, \quad (2.4)$$

$$\langle \tilde{U}_{\mathbf{k}\alpha}(\omega) \tilde{U}_{\mathbf{k}'\beta}(\omega') \rangle = \langle \tilde{U}_{-\mathbf{k}\alpha}^*(-\omega) \tilde{U}_{\mathbf{k}'\beta}(\omega') \rangle = J_{\omega k} \delta_{\alpha\beta}^{\perp} \delta_{\mathbf{k}', -\mathbf{k}} \delta(\omega' + \omega). \quad (2.5)$$

Here and below  $\langle \dots \rangle$  means ensemble average over all realizations of the turbulence,  $\delta_{\mathbf{k}', -\mathbf{k}}$  is the three-dimensional Kronecker symbol (equal to unity only if  $\mathbf{k}' = -\mathbf{k}$ ),

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<sup>2</sup> The helicity is negligible in galaxies on the scales of the smallest turbulent eddy, which is the principal driver of the field evolution [27]. The helicity in protogalaxies is even smaller than that in galaxies.

$\delta(\omega' + \omega)$  is the Dirac  $\delta$ -function,

$$\delta_{\alpha\beta}^{\perp} \stackrel{\text{def}}{=} \delta_{\alpha\beta} - \hat{k}_{\alpha}\hat{k}_{\beta}, \quad (2.6)$$

$\delta_{\alpha\beta}$  is the one-dimensional Kronecker symbol, and  $\hat{\mathbf{k}} = \mathbf{k}/k$  is the unit vector along the wave vector  $\mathbf{k}$ . It is important that function  $J_{\omega k}$ , which stands for the normal (non-helical) part of the turbulence, depends only on the absolute values of  $\omega$  and  $\mathbf{k}$ . Note that equations (2.2) and (2.3) represent the discrete three-dimensional Fourier transformation in space and the continuous one-dimensional Fourier transformation in time (see Appendix A), so that the inverse Fourier transformations are

$$U_{\alpha}(t, \mathbf{r}) = \sum_{\mathbf{k}} \tilde{U}_{\mathbf{k}\alpha}(t) e^{i\mathbf{k}\mathbf{r}} = \left(\frac{L}{2\pi}\right)^3 \int_{-\infty}^{\infty} \tilde{U}_{\alpha}(t, \mathbf{k}) e^{i\mathbf{k}\mathbf{r}} d^3\mathbf{k}, \quad (2.7)$$

$$\tilde{U}_{\mathbf{k}\alpha}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{U}_{\mathbf{k}\alpha}(\omega) e^{-i\omega t} d\omega. \quad (2.8)$$

Here the summation is done over discrete values of vector  $\mathbf{k}$ :  $k_x = (2\pi/L)n_x$ ,  $k_y = (2\pi/L)n_y$ ,  $k_z = (2\pi/L)n_z$ ,  $n_x \in Z$ ,  $n_y \in Z$ ,  $n_z \in Z$ ; and this summation can be replaced by integration of the appropriately defined function  $\tilde{U}_{\alpha}(t, \mathbf{k})$  that is continuous over  $\mathbf{k}$ , see Appendix A. Below we will use both summation and integration over  $\mathbf{k}$ , whichever is more convenient to use.

Let us further assume that the turbulence is Kolmogorov, and that the time correlation function of the turbulent velocities has an exponential profile <sup>3</sup>, i. e. that  $\langle \mathbf{U}(t)\mathbf{U}(t') \rangle \propto e^{-|t-t'|/\tau}$ , where  $\tau$  is the eddy decorrelation time [27], which depends

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<sup>3</sup> Using a Gaussian time correlation profile,  $\langle \mathbf{U}(t)\mathbf{U}(t') \rangle \propto e^{-(t-t')^2/2\tau^2}$ , would be more appropriate. In this case equation (2.10) would become  $J_{\omega k} = J_{0k}e^{-\tau^2\omega^2/2}$ . However, we prefer the exponential profile because it is easier to deal with. (In particular, for Gaussian integrals it is not possible to close integration contours at infinity in the complex plane.)

on  $k$ ,

$$\tau(k) = \tau(0) \left( \frac{k}{k_0} \right)^{-2/3} = \frac{2\pi}{k_0 U_0} \left( \frac{k}{k_0} \right)^{-2/3}. \quad (2.9)$$

Here,  $k_0 = 2\pi/L$  is the smallest wave number of the turbulence, and  $U_0 \sim V_T$  is the largest eddy velocity. In this case we have

$$J_{\omega k} = \frac{J_{0k}}{1 + \tau^2 \omega^2}, \quad (2.10)$$

$$J_{0k} \approx \begin{cases} (U_0/2k_0)(k/k_0)^{-13/3}, & k_0 \leq k \leq k_\nu, \\ 0, & k < k_0, k > k_\nu, \end{cases} \quad (2.11)$$

and, carrying out the inverse Fourier transformations of equation (2.5), in time and in space, we have

$$\langle \tilde{U}_{\mathbf{k}\alpha}(t) \tilde{U}_{\mathbf{k}'\beta}(t') \rangle = \frac{J_{0k}}{2\tau} e^{-|t-t'|/\tau} \delta_{\alpha\beta}^\perp \delta_{\mathbf{k}', -\mathbf{k}}, \quad (2.12)$$

$$\langle U_\alpha(t, \mathbf{r}) U_\beta(t', \mathbf{r}') \rangle = \sum_{\mathbf{k}} \frac{J_{0k}}{2\tau} e^{-|t-t'|/\tau} \delta_{\alpha\beta}^\perp e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}. \quad (2.13)$$

The Kolmogorov kinetic energy spectrum, shown by the thick solid line in Figure 1.4, is [27]

$$I(k) \approx \begin{cases} (U_0^2/k_0)(k/k_0)^{-5/3}, & k_0 \leq k \leq k_\nu, \\ 0, & k < k_0, k > k_\nu, \end{cases} \quad (2.14)$$

so that the total kinetic energy of the fluid motions, per unit mass, is

$$\frac{1}{2} \langle [\mathbf{U}(t, \mathbf{r})]^2 \rangle = \frac{1}{2} \int_{k_0}^{k_\nu} I(k) dk. \quad (2.15)$$

Let us now return to the discussion of the magnetic field energy and of the mag-

netic field structure, the topics which are of most importance in the kinematic dynamo theory. The evolution of the magnetic field energy spectrum in a turbulent conducting plasma was first independently studied by Kazantsev [22], and by Kraichnan and Nagarajan [23] in 1967. Kraichnan and Nagarajan were interested in the limit of small Prandtl numbers, which is not the case in galaxies and protogalaxies. Kazantsev obtained the correct equation for the evolution of the magnetic energy spectrum on small subviscous scales in the limit of large Prandtl numbers. He assumed that the turbulent velocities are  $\delta$ -correlated in time,  $\langle \mathbf{U}(t)\mathbf{U}(t') \rangle \propto \delta(t' - t)$ . In 1982 Vainshtein derived a universal equation for the longitudinal correlation function of magnetic field [51]. He also proved, making only very general assumptions about the turbulence, that the magnetic field is exponentially amplified by the turbulent motions (i. e. that the fast dynamo inductive action does exist). It is interesting that in spite of more general assumptions Vainshtein used, his equation for the magnetic energy spectrum on subviscous scales basically coincides with that of Kazantsev <sup>4</sup>.

The complete correct theory for the evolution of the magnetic field energy spectrum and for the growth of the total magnetic energy in the limit of large Prandtl numbers was first developed by Kulsrud and Anderson in 1992 [27]. They used the kinematic dynamo model and solved equation (2.1) by the quasilinear expansion in time (by iterating it twice in time). Below we list the main results obtained by Kulsrud and Anderson, because in this thesis we will use calculational methods similar to those that they used, and the results we will obtain reduce to their results if the plasma is not magnetized (when the magnetic field is so weak, that  $\omega_i t_i \ll 1$ , and the kinematic dynamo model is valid, see Figure 1.5).

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<sup>4</sup> The equation (25) of Vainshtein and the equation (31) of Kazantsev [22] are the same in the absence of resistivity, except there are different constant factors in front of their right-hand-side parts. The equation (2.25) in this thesis, which was derived by Kulsrud and Anderson [27], would also be the same if converted to the differential equation for function  $k^{-2}M$ .

Kulsrud and Anderson found that the ensemble averaged total magnetic energy per unit mass,  $\mathcal{E} = \langle B^2 \rangle / 8\pi\rho$  ( $\rho$  is the plasma density), grows exponentially in time:

$$\frac{d\mathcal{E}}{dt} = 2\gamma_o\mathcal{E}, \quad (2.16)$$

$$\gamma_o = \frac{1}{3} \sum_{\mathbf{k}} k^2 J_{0k}. \quad (2.17)$$

They defined the magnetic energy spectrum  $M(t, k)$  as

$$M(t, k) \stackrel{\text{def}}{=} \frac{1}{4\pi\rho} \left( \frac{L}{2\pi} \right)^3 \int k^2 \langle |\tilde{\mathbf{B}}(t, \mathbf{k})|^2 \rangle d^2\hat{\mathbf{k}}, \quad (2.18)$$

where the integration is carried out over all directions of  $\hat{\mathbf{k}} = \mathbf{k}/k$ . Here, we consider that the function  $\tilde{\mathbf{B}}_{\mathbf{k}}(t)$  is the Fourier coefficient of the magnetic field  $\mathbf{B}$ , while  $\tilde{\mathbf{B}}(t, \mathbf{k})$  is the appropriately defined function continuous in  $\mathbf{k}$ , so that

$$B_\alpha(t, \mathbf{r}) = \sum_{\mathbf{k}} \tilde{B}_{\mathbf{k}\alpha}(t) e^{i\mathbf{k}\mathbf{r}} = \left( \frac{L}{2\pi} \right)^3 \int_{-\infty}^{\infty} \tilde{B}_\alpha(t, \mathbf{k}) e^{i\mathbf{k}\mathbf{r}} d^3\mathbf{k} \quad (2.19)$$

(see definition of the Fourier transformations and coefficients in Appendix A). The averaged total magnetic energy per unit mass is obviously equal to

$$\mathcal{E} = \frac{1}{2} \int_0^\infty M(t, k) dk. \quad (2.20)$$

Kulsrud and Anderson derived the equation for the evolution of the magnetic energy spectrum  $M(t, k)$  [2, 27]. They called it *the mode coupling equation*, which, in the absence of helicity, is

$$\frac{\partial M}{\partial t} = \int_0^\infty K_o(k, k') M(t, k') dk' - 2 \frac{\eta_{ro}}{4\pi} k^2 M(t, k), \quad (2.21)$$

where <sup>5</sup>

$$K_o(k, k') = 2\pi k^4 \left(\frac{L}{2\pi}\right)^3 \int_0^\pi J_{0k''} \frac{k^2 + k'^2 - kk' \cos \theta}{k''^2} \sin^3 \theta d\theta, \quad (2.22)$$

$$k'' = (k^2 + k'^2 - 2kk' \cos \theta)^{1/2}, \quad (2.23)$$

and  $\theta$  is the angle between  $\mathbf{k}$  and  $\mathbf{k}'$ , see Figure 4.1. The constant  $\eta_{To}$  is the turbulent magnetic diffusivity:

$$\frac{\eta_{To}}{4\pi} = \frac{1}{3} \sum_{\mathbf{k}} J_{0k}. \quad (2.24)$$

Kulsrud and Anderson found that the magnetic energy cascades down to very small subviscous scales via the mode coupling equation (2.21). In particular, this mode coupling equation greatly simplifies for scales much less than the viscous cutoff scale,  $k \gg k_\nu$ , and becomes a simple differential equation, homogeneous in  $k$ ,

$$\frac{\partial M}{\partial t} = \frac{\gamma_o}{5} \left( k^2 \frac{\partial^2 M}{\partial k^2} - 2k \frac{\partial M}{\partial k} + 6M \right). \quad (2.25)$$

If  $M(t, k_{\text{ref}})$  is known as a function of time at some reference wave number  $k = k_{\text{ref}}$ , then the solution of (2.25) is

$$M(t, k) = \int_{-\infty}^t M(t', k_{\text{ref}}) G_o(k/k_{\text{ref}}, t - t') dt', \quad (2.26)$$

where the Green's function  $G_o(k, t)$  is

$$G_o(k, t) = \left(\frac{5}{4\pi}\right)^{1/2} \frac{k^{3/2} \ln k}{\gamma_o^{1/2} t^{3/2}} e^{(3/4)\gamma_o t} e^{-5 \ln^2 k / 4\gamma_o t}. \quad (2.27)$$

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<sup>5</sup> Note, that Kulsrud and Anderson [27] used a slightly different definition of the continuous Fourier transformation in time. As a result, they have additional factors  $2\pi$  in their formulas.

A “signal”  $M(t, k_{\text{ref}})$  at zero time will increase exponentially as  $e^{(3/4)\gamma_0 t}$  and will extend down to the scale  $k_{\text{peak}} \approx e^{\gamma_0 t} k_{\text{ref}}$ , where  $k_{\text{peak}}$  is the peak of function  $kG_o(k, t)$ . As a result, the magnetic energy tends to quickly propagate to very small subviscous scales [2, 27]!

The next important question concerns the magnetic field geometrical structure, which is produced by the kinematic dynamo action. This structure has recently been extensively studied. It was first found that the mean square curvature of the field grows exponentially in time,  $\propto e^{(16/5)\gamma_0 t}$ , and quickly becomes much larger than the square of the viscous wave number  $k_\nu$  [31]. Even if the curvature is initially zero, it starts to grow because of the second order spatial derivatives of the turbulent velocities. Schekochihin *et al.* then found the actual probability distribution for the magnetic field curvature and showed that the curvature stays of the order of the viscous wave number in most of the volume, but becomes extremely large in a small fraction of the volume [46]. Moreover, it turns out that there is an anti-correlation between the value of the curvature and the strength of the magnetic field! Namely, the magnetic field is weak in those (small) regions of the volume, where the field curvature is large, and the field is strong in those (large) regions of the volume, where the curvature is weak. Thus, the curvature stays finite ( $\sim$  the smallest eddy size) over the bulk of the volume, while the magnetic field energy becomes concentrated on very small subviscous scales, as found by Kulsrud and Anderson. Therefore, the field has *the folding structure* with the wave number perpendicular to the field lines much larger than the wave number parallel to the field lines,  $k_\perp \gg k_\parallel \sim k_\nu$ , as demonstrated in Figure 2.1. This folding structure was first suggested by Cowley in 1999 [11] on intuitive grounds and was later supported by MHD numerical simulations. The folding nature of magnetic fields is also theoretically supported by the Barnes

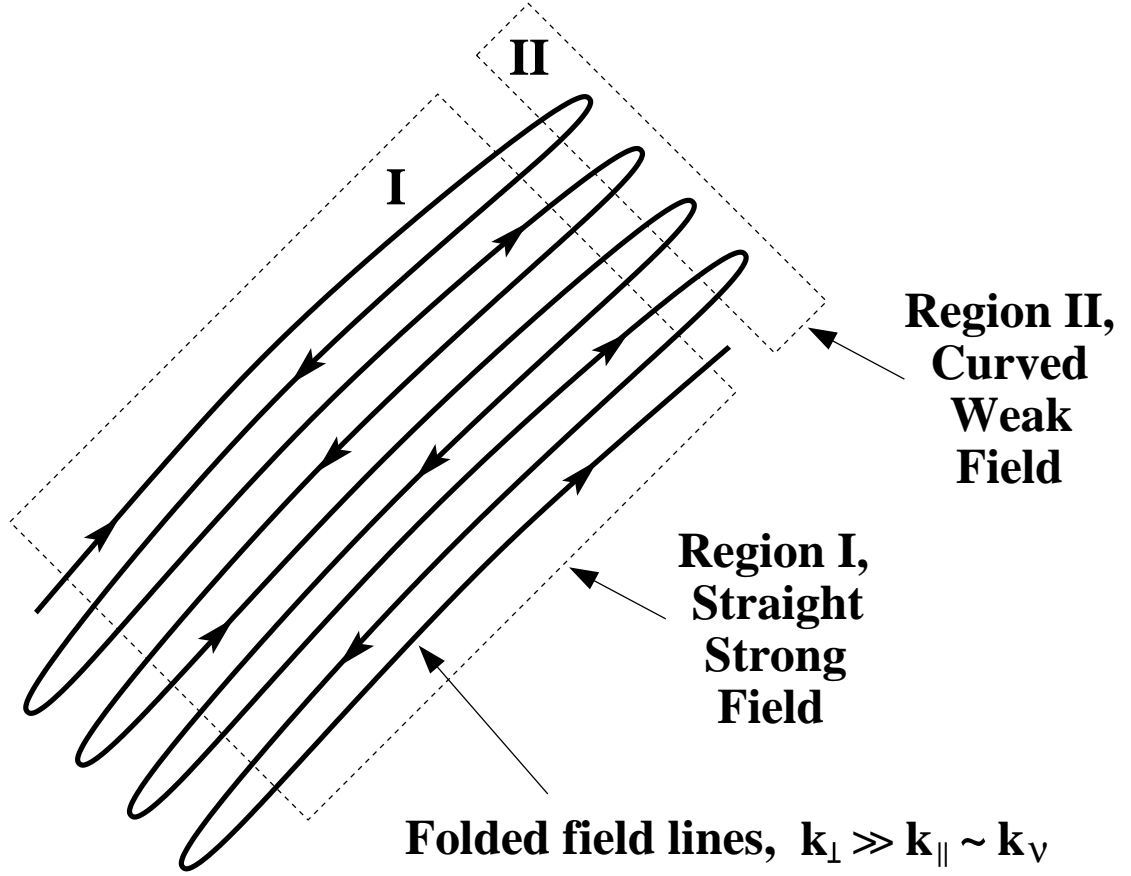


Figure 2.1: The folding structure of the magnetic fields produced by the kinematic dynamo (for simplicity shown in two-dimensions). The field is nearly straight and strong in Region I. The field is very curved but weak in Region II.

collisionless damping [3], which states that small perturbations of the field (MHD waves) are strongly damped by collisionless energy transfer to the electrons <sup>6</sup>. The folding structure of the magnetic field lines is very important for our calculations of the magnetized dynamo!

## 2.2 The Magnetized Turbulent Dynamo

In a protogalaxy the magnetic field is quickly amplified by the kinematic dynamo action. When the field strength becomes of the order of  $\sim 10^{-16}$  G or higher, the ion cyclotron time in the magnetic field becomes less than the ion collision time,  $\omega_i t_i \gg 1$ , (and correspondingly, the ion gyroradius becomes shorter than the ion mean free path). As a result, the plasma becomes strongly magnetized, and the dynamics of the turbulent velocities on the viscous scales starts to be controlled by the magnetic field. These are the scales on which the velocities most rapidly amplify the field (see footnote 2 on page 15). As a result, the main assumption of the kinematic dynamo theory, the absence of the back reaction influence of the field on the turbulence, becomes invalid. The turbulent dynamo becomes what we call the magnetized turbulent dynamo, see Figure 1.5.

It is important to know that the magnetic Lorentz forces can be neglected as long as the magnetic energy stays less than the kinetic energy of the smallest turbulent eddies, i. e. as long as the field strength stays less than  $\sim 10^{-7}$  G in a protogalaxy. During the magnetized dynamo stage in the protogalaxy, when  $\omega_i t_i \gg 1$ , but the magnetic energy is small, see Figure 1.5, the magnetic field strongly affects the turbulent

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<sup>6</sup> It works as follows. Sinusoidal MHD waves slightly perturb the field strength. As a result, the perpendicular kinetic energy of electrons oscillates in time. However, because the electrons reflect from magnetic mirrors, a part of the fluid kinetic energy is irreversibly transferred into the parallel kinetic energy of electrons (it is the second order effect). Thus, the electrons are heated, and the waves are damped.

motions on the viscous scales by changing the viscous forces, while the Lorentz forces are negligible! Indeed, under the condition that  $\omega_i t_i \gg 1$ , the transport of both the transverse and the longitudinal components of ion momentum in the direction perpendicular to the magnetic field lines is inhibited. The transport of the transverse component of ion momentum along magnetic field lines is also inhibited, because gyrating ions quickly lose their “memory” of the transverse ordered velocity in a time equal to the gyrorotation time [7]. The transport of the longitudinal component of ion momentum along the magnetic field lines is the same as in the absence of the field. As a result, the viscous forces acting on turbulent velocities  $\mathbf{V}$  in an incompressible fully ionized plasma are determined by the Braginskii viscosity stress tensor [7]

$$\pi_{\alpha\beta} = -\nu(3\hat{b}_\alpha\hat{b}_\beta - \delta_{\alpha\beta})\hat{b}_\mu\hat{b}_\nu V_{\mu,\nu} , \quad (2.28)$$

where  $\hat{\mathbf{b}} = \mathbf{B}/B$  is the unit vector along the magnetic field. Note, that this stress tensor depends on the field unit vector  $\hat{\mathbf{b}}$ , but is independent of the magnetic strength  $B = |\mathbf{B}|$ , i. e. the viscous forces depend only on the magnetic field direction (as long as the plasma is strongly magnetized, and  $\omega_i t_i \gg 1$ ).

The MHD equations for the turbulent velocities  $\mathbf{V}$  in an incompressible strongly magnetized plasma are [7, 29]

$$\begin{aligned} \partial_t V_\alpha &= -P'_{,\alpha} + f_\alpha - \pi_{\alpha\beta,\beta} - (V_\alpha V_\beta)_{,\beta} \\ &= -P''_{,\alpha} + f_\alpha + 3\nu(\hat{b}_\alpha\hat{b}_\beta\hat{b}_\mu\hat{b}_\nu V_{\mu,\nu})_{,\beta} - (V_\alpha V_\beta)_{,\beta} , \end{aligned} \quad (2.29)$$

$$V_{\alpha,\alpha} = 0, \quad (2.30)$$

where  $\mathbf{f}$  is the force driving the turbulence. In particular, it can be thought as a force, driving a turbulent eddy, that comes from larger eddies. The hydrodynamic

pressure  $P'$  can be determined by the incompressibility condition (2.30). To obtain the second line in equation (2.29), we use formula (2.28) for the viscous stress, and we incorporate the isotropic part of the viscous stress into the pressure  $P''$  (for the same reason the force  $\mathbf{f}$  can be assumed to be solenoidal).

It is difficult to solve equations (2.29) and (2.30) directly because they are non-linear and they include the complicated Braginskii viscous forces. We also do not know the exact expression of the driving force  $\mathbf{f}$ , but its statistics is the same as for an unmagnetized plasma. Therefore, let us proceed as follows. We introduce subsidiary incompressible turbulent velocities  $\mathbf{U}$  which, by definition, satisfy equations

$$\partial_t U_\alpha = -P_{,\alpha}''' + f_\alpha + \frac{1}{5}\nu\Delta U_\alpha - (U_\alpha U_\beta)_{,\beta}, \quad (2.31)$$

$$U_{\alpha,\alpha} = 0 \quad (2.32)$$

( $\Delta U_\alpha = U_{\alpha,\beta\beta}$ , and the pressure  $P'''$  can be determined by the incompressibility condition). Let us analyze and compare equations (2.29) and (2.31).

- First, let us for the moment formally average the Braginskii viscosity term  $3\nu(\hat{b}_\alpha\hat{b}_\beta\hat{b}_\mu\hat{b}_\nu V_{\mu,\nu})_{,\beta}$  in equation (2.29) over all directions of an isotropic magnetic field, i. e. let us apply formula  $\langle\hat{b}_\alpha\hat{b}_\beta\hat{b}_\mu\hat{b}_\nu\rangle_{\mathbf{b}} = (1/15)(\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\mu}\delta_{\beta\nu} + \delta_{\alpha\nu}\delta_{\beta\mu})$  to it. Then this term reduces to  $(1/5)\nu\Delta V_\alpha$ , which coincides with the isotropic viscosity term in equation (2.31). In other words,  $\nu_{\text{eff}} = (1/5)\nu$  could be considered as an effective reduced viscosity for an incompressible fully ionized plasma in the presence of a magnetic field that is isotropically tangled on subviscous scales.
- Second, note that equations (2.29) and (2.31) have the same driving force  $\mathbf{f}$ . By taking the driving force to be the same, we assume that this force comes

from larger turbulent eddies. These larger eddies are on scales larger than the viscous scales, and therefore, these eddies “do not know” whether the viscous forces are of the Braginskii type or of the standard isotropic type.

- Third, note that equation (2.31) is a familiar hydrodynamic equation with a standard isotropic viscosity term. However, it has a reduced molecular viscosity,  $(1/5)\nu$  instead of  $\nu$ . Therefore, we can suppose that the solution of equation (2.31) is the Kolmogorov turbulence with the effective reduced viscosity

$$\nu_{\text{eff}} = \frac{1}{5}\nu. \quad (2.33)$$

As a result, we suppose that the statistical properties and the kinetic energy spectrum of the subsidiary velocities  $\mathbf{U}$  are given by formulas (2.4), (2.5), (2.10)–(2.15), where the Reynolds number  $R$  and the viscous cutoff wave number  $k_\nu$  are now determined by the effective viscosity,  $\nu_{\text{eff}} = (1/5)\nu$ , i. e.  $R \sim V_0 L / \nu_{\text{eff}} = 5V_0 L / \nu$ ,  $k_\nu \sim R^{3/4} k_0 = (5V_0 L / \nu)^{3/4} k_0$ .

- Fourth, we can say that because of the presence of the magnetic field, the Braginskii viscous forces “are doing worse” at dissipating the turbulent motions, as compared with the standard isotropic viscous forces in a field-free plasma. This is why the effective viscosity in equation (2.31) is reduced by a factor  $1/5$ . Thus, we can draw the important conclusion that the spectrum of turbulent velocities in a strongly magnetized plasma extends to a smaller scale,  $2\pi k_\nu^{-1} \sim (5V_0 L / \nu)^{-3/4} L$ , as compared to the cutoff scale in an unmagnetized hydrodynamic plasma,  $2\pi k_\nu^{-1} \sim (V_0 L / \nu)^{-3/4} L$ ! <sup>7</sup>

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<sup>7</sup> This conclusion is important because it means that in a magnetized plasma the smallest turbulent eddies, which are located on the viscous scales, have turnover times shorter than those of the

Now let us subtract equations (2.31) and (2.32) from equations (2.29) and (2.30) to eliminate the unknown driving force  $\mathbf{f}$ , and let us introduce *the back-reaction velocity*  $\mathbf{v} \stackrel{\text{def}}{=} \mathbf{V} - \mathbf{U}$ . We have

$$\begin{aligned} \partial_t v_\alpha &= -P_{,\alpha} + 3\nu(b_{\alpha\beta\mu\nu}v_{\mu,\nu})_{,\beta} + 3\nu(b_{\alpha\beta\mu\nu}U_{\mu,\nu})_{,\beta} - \frac{1}{5}\nu\Delta U_\alpha \\ &\quad - (v_\alpha U_\beta + U_\alpha v_\beta + v_\alpha v_\beta)_{,\beta} , \end{aligned} \quad (2.34)$$

$$v_{\alpha,\alpha} = 0, \quad (2.35)$$

where the pressure  $P = P'' - P'''$ . Here and below we use the following symmetric tensors (which vary in time and in space):

$$b_{\alpha\beta\gamma\delta} \stackrel{\text{def}}{=} \hat{b}_\alpha \hat{b}_\beta \hat{b}_\gamma \hat{b}_\delta , \quad (2.36)$$

$$b_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \hat{b}_\alpha \hat{b}_\beta \hat{b}_\gamma , \quad (2.37)$$

$$b_{\alpha\beta} \stackrel{\text{def}}{=} \hat{b}_\alpha \hat{b}_\beta . \quad (2.38)$$

Velocity  $\mathbf{v}$ , which satisfies equations (2.34) and (2.35), can be considered as the correction to the Kolmogorov velocity  $\mathbf{U}$ . This correction results from the strong influence of the magnetic field direction  $\hat{\mathbf{b}}$  on the turbulent motions through the Braginskii viscosity tensor (2.28). The evolution of the magnetic field  $\mathbf{B}$  is given by the following standard MHD equation [29]

$$\partial_t B_\alpha = V_{\alpha,\beta} B_\beta - V_\beta B_{\alpha,\beta} , \quad (2.39)$$

where the plasma velocities  $\mathbf{V} = \mathbf{U} + \mathbf{v}$  are incompressible [compare with equation (2.29)]. As a result, the magnetic field amplification rate in the magnetized plasma is expected to be higher, compared to that in the unmagnetized plasma. See also footnote 2 on page 15.

tion (2.1)], and we neglect resistivity. Consequently, the equations for the magnetic field squared,  $B^2$ , and for the magnetic unit vector  $\hat{\mathbf{b}}$  are

$$\partial_t B^2 = 2V_{\alpha,\beta} B_\alpha B_\beta - V_\beta (B^2)_{,\beta} , \quad (2.40)$$

$$\partial_t \hat{b}_\alpha = V_{\alpha,\beta} \hat{b}_\beta - V_{\beta,\gamma} b_{\alpha\beta\gamma} - V_\beta \hat{b}_{\alpha,\beta} . \quad (2.41)$$

In the end of this section, it should be noted that the magnetized turbulent dynamo does exist in protogalaxies, but does not in galaxies. The reason for this is that, in contrast to temperatures in protogalaxies, temperatures in galaxies are low enough ( $T \lesssim 10^5$  K) for a considerable number of neutral particles to be present. These neutral particles have mean free path much longer than the ions have. Therefore, the viscosity in galaxies is dominated by the neutral particles, and the viscous forces are given by the standard familiar formula,  $\nu_n \Delta \mathbf{V}$  (here  $\nu_n$  is the neutral viscosity). As a result, in galaxies the turbulence is Kolmogorov and the kinematic dynamo theory can be used<sup>8</sup>.

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<sup>8</sup> Temperatures can be very high in the coronal regions of galaxies, and in these regions the magnetized turbulent dynamo can be present.

## Chapter 3

# Statistics of Turbulent Velocities in Strongly Magnetized Plasmas

The statistics of the turbulent velocities in an unmagnetized plasma is given by formulas (2.4) and (2.5). In this chapter we derive similar equations for the statistics of the turbulent velocities in a strongly magnetized plasma by making use of the quasi-linear (up to the second order) expansion in time, to solve the MHD equations. The final results of the derivations are given by equations (3.80)–(3.83). We then use these results in Chapter 4 to derive formulas for the evolution of the magnetic field total energy and energy spectrum in the magnetized turbulent dynamo theory.

### 3.1 The Quasi-linear Expansion of the MHD Equations

Let us assume that we know the magnetic field at zero time,  $\mathbf{B}|_{t=0} = {}^0\mathbf{B}(\mathbf{r})$  and  $\hat{\mathbf{b}}|_{t=0} = {}^0\hat{\mathbf{b}}(\mathbf{r})$ , and that the back-reaction velocity  $\mathbf{v}$  is initially zero,  $\mathbf{v}|_{t=0} = {}^0\mathbf{v}(\mathbf{r}) \equiv$

0.<sup>1</sup> We advance the magnetic field and the back-reaction velocity to some future time  $t > 0$  by the nonlinear terms, i. e. by integrating equations (2.34), (2.39) and (2.41) twice in time. This quasi-linear expansion procedure is similar to the calculations of Kulsrud and Anderson [27]. Considering  $t$  as the expansion parameter<sup>2</sup>, up to the second order, we have

$$\mathbf{B}(t, \mathbf{r}) = {}^0\mathbf{B}(\mathbf{r}) + {}^1\mathbf{B}(t, \mathbf{r}) + {}^2\mathbf{B}(t, \mathbf{r}), \quad (3.1)$$

$$\hat{\mathbf{b}}(t, \mathbf{r}) = {}^0\hat{\mathbf{b}}(\mathbf{r}) + {}^1\hat{\mathbf{b}}(t, \mathbf{r}) + {}^2\hat{\mathbf{b}}(t, \mathbf{r}), \quad (3.2)$$

$$\mathbf{v}(t, \mathbf{r}) = {}^1\mathbf{v}(t, \mathbf{r}) + {}^2\mathbf{v}(t, \mathbf{r}), \quad (3.3)$$

$$\mathbf{V}(t, \mathbf{r}) = {}^1\mathbf{V}(t, \mathbf{r}) + {}^2\mathbf{V}(t, \mathbf{r}) = [\mathbf{U}(t, \mathbf{r}) + {}^1\mathbf{v}(t, \mathbf{r})] + {}^2\mathbf{v}(t, \mathbf{r}), \quad (3.4)$$

where  $\mathbf{V} = \mathbf{U} + \mathbf{v}$  is the total fluid velocity, and the incompressible Kolmogorov turbulent velocities  $\mathbf{U}$  are considered to be given and to be of the first order [27, 50]. All first and second order quantities, except  $\mathbf{U}$ , are initially zero, e. g.  ${}^1\mathbf{B}(0, \mathbf{r}) = 0$  and  ${}^1\mathbf{v}(0, \mathbf{r}) = 0$ .

Now, we substitute the above expansion formulas into equations (2.39), (2.41) and into equations (2.34)–(2.37). We find that the zero order equations are

$$\partial_t {}^0B_\alpha = 0, \quad (3.5)$$

$$\partial_t {}^0\hat{b}_\alpha = 0, \quad (3.6)$$

$${}^0v_\alpha = 0, \quad (3.7)$$

$${}^0V_\alpha = 0, \quad (3.8)$$

$${}^0b_{\alpha\beta\gamma} = {}^0\hat{b}_\alpha {}^0\hat{b}_\beta {}^0\hat{b}_\gamma, \quad (3.9)$$

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<sup>1</sup> A nonzero initial back-reaction velocity would lead to transients which would be dissipated anyway.

<sup>2</sup> If we would like to be more formal, then we need to introduce a dimensionless variable  $\xi = t/\Delta t$ , and to consider  $U_{\alpha,\beta}\Delta t$  and  $V_{\alpha,\beta}\Delta t$ , which are dimensionless, as the expansion parameters.

$${}^0b_{\alpha\beta\mu\nu} = {}^0\hat{b}_\alpha {}^0\hat{b}_\beta {}^0\hat{b}_\mu {}^0\hat{b}_\nu ; \quad (3.10)$$

the first order equations are

$$\partial_t {}^1B_\alpha = {}^1V_{\alpha,\beta} {}^0B_\beta - {}^1V_\beta {}^0B_{\alpha,\beta} , \quad (3.11)$$

$$\partial_t {}^1\hat{b}_\alpha = {}^1V_{\alpha,\beta} {}^0\hat{b}_\beta - {}^1V_{\beta,\gamma} {}^0b_{\alpha\beta\gamma} - {}^1V_\beta {}^0\hat{b}_{\alpha,\beta} , \quad (3.12)$$

$$\partial_t {}^1v_\alpha = -{}^1P_{,\alpha} + 3\nu({}^0b_{\alpha\beta\mu\nu} {}^1v_{\mu,\nu})_{,\beta} + 3\nu({}^0b_{\alpha\beta\mu\nu} U_{\mu,\nu})_{,\beta} - \frac{1}{5}\nu\Delta U_\alpha , \quad (3.13)$$

$${}^1v_{\alpha,\alpha} = 0, \quad (3.14)$$

$${}^1V_\alpha = U_\alpha + {}^1v_\alpha , \quad (3.15)$$

$${}^1b_{\alpha\beta\gamma} = {}^1\hat{b}_\alpha {}^0\hat{b}_\beta {}^0\hat{b}_\gamma + {}^0\hat{b}_\alpha {}^1\hat{b}_\beta {}^0\hat{b}_\gamma + {}^0\hat{b}_\alpha {}^0\hat{b}_\beta {}^1\hat{b}_\gamma , \quad (3.16)$$

$${}^1b_{\alpha\beta\mu\nu} = {}^1\hat{b}_\alpha {}^0\hat{b}_\beta {}^0\hat{b}_\mu {}^0\hat{b}_\nu + {}^0\hat{b}_\alpha {}^1\hat{b}_\beta {}^0\hat{b}_\mu {}^0\hat{b}_\nu + {}^0\hat{b}_\alpha {}^0\hat{b}_\beta {}^1\hat{b}_\mu {}^0\hat{b}_\nu + {}^0\hat{b}_\alpha {}^0\hat{b}_\beta {}^0\hat{b}_\mu {}^1\hat{b}_\nu ; \quad (3.17)$$

and the second order equations are

$$\partial_t {}^2B_\alpha = {}^1V_{\alpha,\beta} {}^1B_\beta + {}^2V_{\alpha,\beta} {}^0B_\beta - {}^1V_\beta {}^1B_{\alpha,\beta} - {}^2V_\beta {}^0B_{\alpha,\beta} , \quad (3.18)$$

$$\begin{aligned} \partial_t {}^2\hat{b}_\alpha &= {}^1V_{\alpha,\beta} {}^1\hat{b}_\beta + {}^2V_{\alpha,\beta} {}^0\hat{b}_\beta - {}^1V_{\beta,\gamma} {}^1b_{\alpha\beta\gamma} - {}^2V_{\beta,\gamma} {}^0b_{\alpha\beta\gamma} \\ &\quad - {}^1V_\beta {}^1\hat{b}_{\alpha,\beta} - {}^2V_\beta {}^0\hat{b}_{\alpha,\beta} , \end{aligned} \quad (3.19)$$

$$\begin{aligned} \partial_t {}^2v_\alpha &= -{}^2P_{,\alpha} + \left( 3\nu {}^1b_{\alpha\beta\mu\nu} {}^1v_{\mu,\nu} + 3\nu {}^0b_{\alpha\beta\mu\nu} {}^2v_{\mu,\nu} + 3\nu {}^1b_{\alpha\beta\mu\nu} U_{\mu,\nu} \right. \\ &\quad \left. - {}^1v_\alpha U_\beta - U_\alpha {}^1v_\beta - {}^1v_\alpha {}^1v_\beta \right)_{,\beta} , \end{aligned} \quad (3.20)$$

$${}^2v_{\alpha,\alpha} = 0, \quad (3.21)$$

$${}^2V_\alpha = {}^2v_\alpha . \quad (3.22)$$

Here, of course, the pressure is also expanded,  $P = {}^1P + {}^2P$ , and it is completely determined by the incompressibility conditions (3.14) and (3.21).

## 3.2 The Time Laplace Transform Solution for the Velocities

Let us first solve the first order equations (3.13)–(3.15) to find the first order back-reaction velocity  ${}^1\mathbf{v}$  and the first order total velocity  ${}^1\mathbf{V}$ . It is convenient to Fourier transform these equations in space,  $\mathbf{r} \rightarrow \mathbf{k}$ , by making use of the discrete Fourier transformation, and to Laplace transform them in time,  $t \rightarrow s$ , by making use of the continuous Laplace transformation (see Appendix A). We have

$$s^1\tilde{v}_{\mathbf{k}\alpha}(s) = -ik_\alpha {}^1\tilde{P}_{\mathbf{k}}(s) + 3\nu ik_\beta \sum_{\substack{\mathbf{k}' \\ \mathbf{k}''=\mathbf{k}-\mathbf{k}'}} ik'_\nu {}^1\tilde{v}_{\mathbf{k}'\mu}(s) {}^0\tilde{b}_{\mathbf{k}''\alpha\beta\mu\nu} \\ + 3\nu ik_\beta \sum_{\substack{\mathbf{k}' \\ \mathbf{k}''=\mathbf{k}-\mathbf{k}'}} ik'_\nu \tilde{U}_{\mathbf{k}'\mu}(s) {}^0\tilde{b}_{\mathbf{k}''\alpha\beta\mu\nu} + \frac{1}{5}\nu k^2 \tilde{U}_{\mathbf{k}\alpha}(s), \quad (3.23)$$

$$k_\alpha {}^1\tilde{v}_{\mathbf{k}\alpha}(s) = 0, \quad (3.24)$$

$${}^1\tilde{V}_{\mathbf{k}\alpha}(s) = \tilde{U}_{\mathbf{k}\alpha}(s) + {}^1\tilde{v}_{\mathbf{k}\alpha}(s), \quad (3.25)$$

where  ${}^1\tilde{v}_{\mathbf{k}\alpha}$ ,  ${}^1\tilde{P}_{\mathbf{k}}$ ,  ${}^1\tilde{v}_{\mathbf{k}'\mu}$ ,  $\tilde{U}_{\mathbf{k}'\mu}$ ,  $\tilde{U}_{\mathbf{k}\alpha}$  and  ${}^1\tilde{V}_{\mathbf{k}\alpha}$  are the Fourier-Laplace coefficients, and  ${}^0\tilde{b}_{\mathbf{k}''\alpha\beta\mu\nu}$  is the Fourier coefficient ( ${}^0\tilde{b}_{\mathbf{k}''\alpha\beta\mu\nu}$  is constant in time because it is of the zero order). In the derivation of equation (3.23) we also make use of the zero initial condition,  ${}^1\mathbf{v}|_{t=0} = 0$  [see equation (A.11) in Appendix A]. Now, we multiply equation (3.23) on the left by tensor  $\delta_{\gamma\alpha}^\perp = \delta_{\gamma\alpha} - \hat{k}_\gamma \hat{k}_\alpha$ , which is perpendicular to the unit vector  $\hat{\mathbf{k}} = \mathbf{k}/k$ , see equation (2.6). The pressure term goes away. Using the incompressibility condition (3.24), interchanging indices, and using the symmetry property of tensor  ${}^0\tilde{b}_{\mathbf{k}''\alpha\beta\mu\nu}$  over indices  $\alpha$ ,  $\beta$ ,  $\mu$ , and  $\nu$ , we obtain

$${}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta} {}^1\tilde{v}_{\mathbf{k}'\beta} = {}^1\mathcal{F}_{\mathbf{k}\alpha}, \quad (3.26)$$

$${}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta} = s\delta_{\mathbf{k}\alpha, \mathbf{k}'\beta} + 3\nu\delta_{\alpha\gamma}^\perp k_\mu k'_\nu {}^0\tilde{b}_{\mathbf{k}''\gamma\mu\nu\beta}, \quad \mathbf{k}'' = \mathbf{k} - \mathbf{k}', \quad (3.27)$$

$${}^1\mathcal{F}_{\mathbf{k}\alpha} = \left[ -{}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta} + \left( s + \frac{1}{5}\nu k^2 \right) \delta_{\mathbf{k}\alpha, \mathbf{k}'\beta} \right] \tilde{U}_{\mathbf{k}'\beta} . \quad (3.28)$$

Here we use convenient matrix notation, so that summation is implicitly assumed not only over repeated spatial indices but also over repeated wave numbers [e. g. a summation over  $\beta = x, y, z$  and an infinite summation over all discrete values of  $\mathbf{k}'$  are assumed on the left-hand-side of equation (3.26)]. Function  $\delta_{\mathbf{k}\alpha, \mathbf{k}'\beta} = \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\alpha\beta}$  is the Kronecker tensor, it is equal to unity if  $\alpha = \beta$ ,  $\mathbf{k} = \mathbf{k}'$ , and is zero otherwise. The matrix operator  ${}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta}(s)$  is of the zero order, while “the driving force”  ${}^1\mathcal{F}_{\mathbf{k}\alpha}(s)$  is of the first order.

If there exist matrix  ${}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta}^{-1}$  that is inverse to the matrix  ${}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta}$ , then the solution of equation (3.26) is

$${}^1\tilde{U}_{\mathbf{k}\alpha} = {}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta}^{-1} {}^1\mathcal{F}_{\mathbf{k}'\beta} . \quad (3.29)$$

Using this formula and equations (3.25), (3.28), we obtain the Fourier-Laplace coefficient of the first order total velocity  ${}^1\mathbf{V}$ ,

$${}^1\tilde{V}_{\mathbf{k}\alpha}(s) = \tilde{U}_{\mathbf{k}\alpha} + {}^1\tilde{U}_{\mathbf{k}\alpha} = \left( s + \frac{1}{5}\nu k^2 \right) {}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta}^{-1} \tilde{U}_{\mathbf{k}'\beta}(s) . \quad (3.30)$$

Now, let us consider an important case for which we can invert matrix  ${}^0\mathcal{M}_{\mathbf{k}\alpha, \mathbf{k}'\beta}$ , given by formula (3.27). Let  $({}^0\mathbf{B} \cdot \nabla) {}^0\mathbf{B} = 0$ , i. e. let the magnetic field at zero time,  ${}^0\mathbf{B}$ , vary only in the direction perpendicular to itself. This is equivalent to the case when the magnetic field lines are initially straight, the initial curvature of the field is zero, and  ${}^0b_{\alpha\beta} = {}^0\hat{b}_\alpha {}^0\hat{b}_\beta \equiv \text{const}$ . This model of straight magnetic field lines is not as artificial as it seems at first glance, because of the following arguments. Remember, that in the kinematic dynamo case, the magnetic field lines have the folding pattern,

shown in Figure 2.1. This folding pattern implies that in the bulk of the volume the field is strong and has small curvature,  $k_{\perp} \gg k_{\parallel} \sim k_{\nu}$ , while in a small fraction of the volume the field is weak and curved,  $k_{\perp} \sim k_{\parallel} \gg k_{\nu}$ . The regions of weak and curved field, Region II in Figure 2.1, can be disregarded as long as we consider the volume averaged magnetic field energy spectrum and are not interested in the field curvature. As for the regions of strong magnetic field with small curvature, Region I in Figure 2.1, the field in these regions can be well approximated by our straight magnetic field lines model on scales  $k$  which satisfy  $k \gtrsim k_{\parallel} \sim k_{\nu}$ . However, it should be made clear that we make an assumption that in the magnetized turbulent dynamo case the magnetic field has a folding structure similar to that in the kinematic turbulent dynamo case. This assumption is our *first working hypothesis*. We do not prove it, but there exist a simple argument by contradiction, supporting it, which is as follows. If the magnetic field were not folded, i. e. if  $k_{\perp} \sim k_{\parallel} \gtrsim k_{\nu}$  everywhere in space, then the field would be isotropically tangled on viscous and subviscous scales. In this case the Braginskii viscosity could be averaged over the field directions, and it would probably rapidly reduce to the isotropically averaged viscosity  $\nu_{\text{eff}} = \nu/5$ , see equation (2.33). As a result, in this case the magnetized dynamo would probably possess the properties of the kinematic dynamo with this effective viscosity, and would develop the folding structure of the magnetic field.

In the case when  ${}^0b_{\alpha\beta} \equiv \text{const}$ , we have  ${}^0\tilde{b}_{\mathbf{k}'',\gamma\mu\nu\beta} = \delta_{\mathbf{k}'',0} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\mu} {}^0\hat{b}_{\nu} {}^0\hat{b}_{\beta}$ , and matrix (3.27) reduces to

$${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta} = \delta_{\mathbf{k},\mathbf{k}'} \left[ s\delta_{\alpha\beta} + 3\nu k^2 \mu^2 \delta_{\alpha\gamma}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\beta} \right], \quad (3.31)$$

$$\mu \stackrel{\text{def}}{=} {}^0\hat{\mathbf{b}} \cdot \hat{\mathbf{k}}. \quad (3.32)$$

Here  $\mu$  is the cosine of the angle between vectors  ${}^0\hat{\mathbf{b}}$  and  $\hat{\mathbf{k}}$ . The matrix  ${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta}$  is

now diagonal in  $\mathbf{k}$ , so we can easily invert it:

$${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta}^{-1} = \frac{\delta_{\mathbf{k},\mathbf{k}'}}{s} \left[ \delta_{\alpha\beta} - \frac{3\nu k^2 \mu^2 (1 - \mu^2)}{s + 3\nu k^2 \mu^2 (1 - \mu^2)} \frac{\delta_{\alpha\gamma}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\beta}}{1 - \mu^2} \right]. \quad (3.33)$$

Next, we substitute this inverse matrix into equation (3.30) and obtain the Fourier-Laplace coefficient of the first order total velocity

$${}^1\tilde{V}_{\mathbf{k}\alpha}(s) = {}^1\tilde{V}'_{\mathbf{k}\alpha}(s) + {}^1\tilde{V}''_{\mathbf{k}\alpha}(s), \quad (3.34)$$

$${}^1\tilde{V}'_{\mathbf{k}\alpha}(s) = \frac{s + \bar{\Omega}}{s} \left[ \delta_{\alpha\beta} - \frac{\delta_{\alpha\gamma}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\beta}}{1 - \mu^2} \right] \tilde{U}_{\mathbf{k}\beta}(s), \quad (3.35)$$

$${}^1\tilde{V}''_{\mathbf{k}\alpha}(s) = \frac{s + \bar{\Omega}}{s + 2\Omega} \frac{\delta_{\alpha\gamma}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\beta}}{1 - \mu^2} \tilde{U}_{\mathbf{k}\beta}(s), \quad (3.36)$$

where, we introduce the following notations for the viscous damping frequencies

$$\bar{\Omega} \stackrel{\text{def}}{=} \frac{1}{5} \nu k^2 = \nu_{\text{eff}} k^2, \quad (3.37)$$

$$\Omega \stackrel{\text{def}}{=} \frac{3}{2} \nu k^2 \mu^2 (1 - \mu^2) = \frac{15}{2} \bar{\Omega} \mu^2 (1 - \mu^2). \quad (3.38)$$

The frequency  $\bar{\Omega}$  represents the effective averaged rate of the Braginskii viscous dissipation, see equation (2.33). The frequency  $\Omega$  depends on  $\mu^2 = ({}^0\hat{\mathbf{b}}\hat{\mathbf{k}})^2$ , and this dependence reflects the anisotropy of the Braginskii viscous stress tensor. Note, that  $\bar{\Omega}$  is equal to  $\Omega$  averaged over  $\mu$ .

Now, we calculate  $\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) \rangle$  and  $\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{\mathbf{k}'\beta}(t') \rangle$ , which are the ensemble averages of the  $\mathbf{V}$ 's over all possible realizations of the turbulent motions. For this, first, we need the ensemble averages of the  $\mathbf{U}$ 's. We use the Laplace and Fourier

transformation formulas (see Appendix A), and equations (2.4), (2.5) to obtain

$$\begin{aligned}\langle \tilde{U}_{\mathbf{k}\alpha}(s) \rangle &= \int_0^\infty \langle \tilde{U}_{\mathbf{k}\alpha}(t) \rangle e^{-st} dt \\ &= \int_0^\infty e^{-st} dt \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \langle \tilde{U}_{\mathbf{k}\alpha}(\omega) \rangle e^{-i\omega t} d\omega = 0,\end{aligned}\tag{3.39}$$

$$\begin{aligned}\langle \tilde{U}_{\mathbf{k}\alpha}(s) \tilde{U}_{\mathbf{k}'\beta}(s') \rangle &= \int_0^\infty \int_0^\infty \langle \tilde{U}_{\mathbf{k}\alpha}(t) \tilde{U}_{\mathbf{k}'\beta}(t') \rangle e^{-st-s't'} dt' dt \\ &= \int_0^\infty \int_0^\infty e^{-st-s't'} dt' dt \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty \langle \tilde{U}_{\mathbf{k}\alpha}(\omega) \tilde{U}_{\mathbf{k}'\beta}(\omega') \rangle e^{-i\omega t - i\omega' t'} d\omega' d\omega \\ &= \delta_{\alpha\beta}^\perp \delta_{\mathbf{k}', -\mathbf{k}} \frac{1}{2\pi} \int_{-\infty}^\infty J_{\omega k} d\omega \int_0^\infty \int_0^\infty e^{-st-i\omega t} e^{-s't'+i\omega t'} dt' dt \\ &= \delta_{\alpha\beta}^\perp \delta_{\mathbf{k}', -\mathbf{k}} \frac{1}{2\pi} \int_{-\infty}^\infty J_{\omega k} \frac{d\omega}{(s+i\omega)(s'-i\omega)}.\end{aligned}\tag{3.40}$$

Second, we use these formulas to calculate the ensemble averages of the two modes of the velocity Fourier-Laplace coefficient in equation (3.34),  ${}^1\tilde{V}'_{\mathbf{k}\alpha}(s)$  and  ${}^1\tilde{V}''_{\mathbf{k}\alpha}(s)$ .

Using equations (3.35) and (3.36), we obtain

$$\langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(s) \rangle = 0,\tag{3.41}$$

$$\langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(s) \rangle = 0,\tag{3.42}$$

$$\langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(s) {}^1\tilde{V}''_{\mathbf{k}'\beta}(s') \rangle = 0,\tag{3.43}$$

$$\begin{aligned}\langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(s) {}^1\tilde{V}'_{\mathbf{k}'\beta}(s') \rangle &= \delta_{\mathbf{k}', -\mathbf{k}} \frac{(s+\bar{\Omega})(s'+\bar{\Omega})}{ss'} \left[ \delta_{\alpha\beta}^\perp - \frac{\delta_{\alpha\gamma}^\perp \delta_{\beta\tau}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\tau}{1-\mu^2} \right] \\ &\quad \times \frac{1}{2\pi} \int_{-\infty}^\infty \frac{J_{\omega k} d\omega}{(s+i\omega)(s'-i\omega)},\end{aligned}\tag{3.44}$$

$$\begin{aligned}\langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(s) {}^1\tilde{V}''_{\mathbf{k}'\beta}(s') \rangle &= \delta_{\mathbf{k}', -\mathbf{k}} \frac{(s+\bar{\Omega})(s'+\bar{\Omega})}{(s+2\Omega)(s'+2\Omega)} \frac{\delta_{\alpha\gamma}^\perp \delta_{\beta\tau}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\tau}{1-\mu^2} \\ &\quad \times \frac{1}{2\pi} \int_{-\infty}^\infty \frac{J_{\omega k} d\omega}{(s+i\omega)(s'-i\omega)}.\end{aligned}\tag{3.45}$$

Third, using these last equations and equation (3.34), we easily obtain

$$\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(s) \rangle = 0, \quad (3.46)$$

$$\begin{aligned} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(s) {}^1\tilde{V}_{\mathbf{k}'\beta}(s') \rangle &= \langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(s) {}^1\tilde{V}'_{\mathbf{k}'\beta}(s') \rangle + \langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(s) {}^1\tilde{V}''_{\mathbf{k}'\beta}(s') \rangle \\ &= \delta_{\mathbf{k}', -\mathbf{k}} \frac{1}{2\pi} \int_{-\infty}^{\infty} J_{\omega k} d\omega \\ &\quad \times \left[ \tilde{H}_L(s; \omega, 0) \tilde{H}_L(s'; -\omega, 0) \left( \delta_{\alpha\beta}^{\perp} - \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right) \right. \\ &\quad \left. + \tilde{H}_L(s; \omega, \Omega) \tilde{H}_L(s'; -\omega, \Omega) \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right], \end{aligned} \quad (3.47)$$

where

$$\tilde{H}_L(p; q_1, q_2) = \frac{p + \bar{\Omega}}{(p + iq_1)(p + 2q_2)}. \quad (3.48)$$

The inverse Laplace transformation  $p \rightarrow t$  of function  $\tilde{H}_L(p; q_1, q_2)$  is

$$\begin{aligned} H_L(t, q_1, q_2) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{H}_L(p; q_1, q_2) e^{tp} dp \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{p + \bar{\Omega}}{(p + iq_1)(p + 2q_2)} e^{tp} dp \\ &= \frac{(\bar{\Omega} - iq_1)e^{-iq_1 t} - (\bar{\Omega} - 2q_2)e^{-2q_2 t}}{2q_2 - iq_1}, \end{aligned} \quad (3.49)$$

where we carry out the integral by closing the integration contour in the complex plane and by calculating the residues at  $p = -iq_1, -2q_2$ . Next, we apply the inverse Laplace transformations  $s \rightarrow t$  and  $s' \rightarrow t'$  to equations (3.46) and (3.47). We finally

obtain the desired results

$$\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) \rangle = 0, \quad (3.50)$$

$$\begin{aligned} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{\mathbf{k}'\beta}(t') \rangle &= \langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(t) {}^1\tilde{V}'_{\mathbf{k}'\beta}(t') \rangle + \langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(t) {}^1\tilde{V}''_{\mathbf{k}'\beta}(t') \rangle \\ &= \delta_{\mathbf{k}', -\mathbf{k}} \frac{1}{2\pi} \int_{-\infty}^{\infty} J_{\omega k} d\omega \\ &\quad \times \left[ H_L(t; \omega, 0) H_L(t'; -\omega, 0) \left( \delta_{\alpha\beta}^{\perp} - \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right) \right. \\ &\quad \left. + H_L(t; \omega, \Omega) H_L(t'; -\omega, \Omega) \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right], \quad (3.51) \end{aligned}$$

where function  $H_L(t, q_1, q_2)$  is given by equation (3.49). As one might expect, the ensemble averaged first order turbulent velocity,  $\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) \rangle$ , is zero<sup>3</sup>, while the averaged velocity correlation tensor,  $\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{\mathbf{k}'\beta}(t') \rangle$  is not zero because it is of the second order.

So far we have considered only the first order velocity,  ${}^1\mathbf{V}$ , and have found its statistics, given by equations (3.50) and (3.51). In order to find the second order velocity,  ${}^2\mathbf{V}$ , we need to solve the complicated second order equations (3.20)–(3.22). Fortunately, we will need only the ensemble average of the second order velocity,  $\langle {}^2\tilde{V}_{\mathbf{k}\alpha}(t) \rangle$ . It turns out that in our case of a straight initial field,  ${}^0b_{\alpha\beta} = \text{const}$ , which we consider here, this average is zero,

$$\langle {}^2\tilde{V}_{\mathbf{k}\alpha}(t) \rangle = 0, \quad (3.52)$$

see Appendix B. The reason for this simple result is that  ${}^0b_{\alpha\beta}$  is constant in space, and

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<sup>3</sup> This is because there is no preferred direction. Of course, there is a preferred axis in space, which is along the magnetic field unit vector. However, there is no preferred direction because the Braginskii viscous stress tensor (2.28) is invariant with respect to a substitution  $\hat{\mathbf{b}} \rightarrow -\hat{\mathbf{b}}$  (gyrating ions “do not care” about the exact direction of  $\hat{\mathbf{b}}$ ).

the hydrodynamic turbulence  $\mathbf{U}$  is homogeneous. Therefore, the ensemble average of the term in the parentheses in equation (3.20) is constant in space, and its spatial derivative is zero. As a result, the ensemble averaged velocity  $\langle {}^2\tilde{V}_{\mathbf{k}\alpha}(t) \rangle = \langle {}^2\tilde{v}_{\mathbf{k}\alpha}(t) \rangle$  is zero (see Appendix B for the proof).

Now, let us return to the first order velocities,  ${}^1\mathbf{V}$ . According to equation (3.50) the ensemble average of  ${}^1\mathbf{V}$  is zero,  $\langle {}^1\mathbf{V} \rangle = 0$ . Let us calculate the ensemble average of  ${}^1\mathbf{V}$  squared. We have

$$\begin{aligned} \langle {}^1\mathbf{V}^2 \rangle &= \langle {}^1V_\alpha(t, \mathbf{r}) {}^1V_\alpha(t, \mathbf{r}) \rangle = \sum_{\mathbf{k}, \mathbf{k}'} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{\mathbf{k}'\alpha}(t) \rangle e^{i(\mathbf{k}+\mathbf{k}')\mathbf{r}} \\ &= \sum_{\mathbf{k}} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{-\mathbf{k}\alpha}(t) \rangle = \sum_{\mathbf{k}} \langle |{}^1\tilde{\mathbf{V}}_{\mathbf{k}}(t)|^2 \rangle. \end{aligned} \quad (3.53)$$

Here we use the fact that  $\langle {}^1\tilde{V}_{\mathbf{k}\alpha} {}^1\tilde{V}_{\mathbf{k}'\beta} \rangle = \langle {}^1\tilde{V}_{\mathbf{k}\alpha} {}^1\tilde{V}_{-\mathbf{k}'\beta}^* \rangle \propto \delta_{\mathbf{k}', -\mathbf{k}}$ , see equation (3.51).

Using equations (3.49) and (3.51), we obtain

$$\begin{aligned} \langle |{}^1\tilde{\mathbf{V}}_{\mathbf{k}}(t)|^2 \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} J_{\omega k} d\omega [H_L(t; \omega, 0)H_L(t; -\omega, 0) + H_L(t; \omega, \Omega)H_L(t; -\omega, \Omega)] \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} J_{\omega k} d\omega \left\{ 2\bar{\Omega}^2 \frac{1 - \cos \omega t}{\omega^2} + 2\bar{\Omega} \frac{\sin \omega t}{\omega} + 1 \right. \\ &\quad \left. + \frac{[\bar{\Omega} - (\bar{\Omega} - 2\Omega)e^{-2\Omega t}]^2}{\omega^2 + 4\Omega^2} + 2\bar{\Omega}(\bar{\Omega} - 2\Omega)e^{-2\Omega t} \frac{1 - \cos \omega t}{\omega^2 + 4\Omega^2} \right. \\ &\quad \left. + 2(\bar{\Omega} - 2\Omega)e^{-2\Omega t} \frac{\omega \sin \omega t}{\omega^2 + 4\Omega^2} + \frac{\omega^2}{\omega^2 + 4\Omega^2} \right\}. \end{aligned} \quad (3.54)$$

Let us analyze this last equation. The width of function  $J_{\omega k}$  in  $\omega$  space is about  $1/\tau(k)$ , where  $\tau(k)$  is the decorrelation time of hydrodynamic eddies on scale  $2\pi k^{-1}$ , e. g. see equation (2.10). We have an obvious estimate  $1/\tau(k) \lesssim 1/\tau(k_\nu) \sim \nu_{\text{eff}} k_\nu^2$ , where  $k_\nu$  is the viscous cutoff wave number, and  $\nu_{\text{eff}}$  is the effective viscosity (2.33). In order to make an estimate of the right-hand-side of equation (3.54), let us for the

moment consider the case when the angle between  $\hat{\mathbf{k}}$  and  ${}^0\hat{\mathbf{b}}$  is neither very close to zero or  $\pi$ , nor very close to  $\pm\pi/2$ , so that  $2\Omega = 15\bar{\Omega}\mu^2(1-\mu^2)$  is not small ( $\mu = \hat{\mathbf{k}} \cdot {}^0\hat{\mathbf{b}}$ ). Then, in the limit  $t \gg \tau$  we have  $\Omega t \gg 1$ ,  $\bar{\Omega}t \gg 1$  and  $\omega^{-1} \sin \omega t \sim \pi \delta(\omega)$ . Therefore, we can drop the exponential terms in equation (3.54), and the first term in the brackets  $\{\dots\}$  in this equation gives the main contribution to the integral over  $\omega$ . As a result, equation (3.54) reduces to

$$\langle |{}^1\tilde{\mathbf{V}}_{\mathbf{k}}(t)|^2 \rangle \approx J_{0k}\bar{\Omega}^2 t \propto t, \quad t \gg \tau. \quad (3.55)$$

The velocity modes, which have the angle between  $\hat{\mathbf{k}}$  and  ${}^0\hat{\mathbf{b}}$  not close to 0,  $\pi$ , or  $\pm\pi/2$ , grow in time <sup>4</sup>. Because all terms under the sum in equation (3.53) are nonnegative, the ensemble average of the first order velocity squared,  $\langle {}^1\mathbf{V}^2 \rangle$ , also grows in time. It is clear that in reality this growth can not be as unrestricted, as it appears to be, according to equations (3.53) and (3.55). In the next section we will discuss this growth in more details. We will see that this growth is a feature of the anisotropy of the Braginskii viscous forces and of our quasi-linear expansion procedure, and is indeed restricted.

### 3.3 The Effective Rotational Damping of Velocities

Let us try to understand the reasons of the growth of the ensemble averaged first order velocity squared,  $\langle {}^1\mathbf{V}^2 \rangle$ , which we found in the previous section. First, refer to equations (3.43)–(3.45) and equations (3.47), (3.51). From calculations of the first

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<sup>4</sup> If the angle between  $\hat{\mathbf{k}}$  and  ${}^0\hat{\mathbf{b}}$  is 0,  $\pi$ , or  $\pm\pi/2$ , then  $\Omega = 0$ . In this case equation (3.54) obviously reduces to  $\langle |{}^1\tilde{\mathbf{V}}_{\mathbf{k}}(t)|^2 \rangle = 2J_{0k}\bar{\Omega}^2 t \propto t$ , and the growth is even faster.

order velocity growth, which follow equation (3.51), it is clear that in the case when the angle between unit vectors  $\hat{\mathbf{k}}$  and  ${}^0\hat{\mathbf{b}}$  is not equal to 0,  $\pi$  or  $\pm\pi/2$ , the growth of  ${}^1\mathbf{V}(t) = {}^1\mathbf{V}'(t) + {}^1\mathbf{V}''(t)$  happens due to the growth of the  ${}^1\tilde{V}'_{\mathbf{k}\alpha}(t)$  modes, while the  ${}^1\tilde{V}''_{\mathbf{k}\alpha}(t)$  modes do not grow unrestrictively<sup>5</sup>. Next, let us refer to equations (3.35) and (3.36) for  ${}^1\tilde{V}'_{\mathbf{k}\alpha}$  and  ${}^1\tilde{V}''_{\mathbf{k}\alpha}$ . On one hand, we have

$$\hat{k}_\alpha {}^1\tilde{V}'_{\mathbf{k}\alpha}(s) = 0, \quad (3.56)$$

$$\hat{k}_\alpha {}^1\tilde{V}''_{\mathbf{k}\alpha}(s) = 0, \quad (3.57)$$

as it should be because plasma velocities are incompressible. On the other hand, we have

$${}^0\hat{b}_\alpha {}^1\tilde{V}'_{\mathbf{k}\alpha}(s) = 0, \quad (3.58)$$

$${}^0\hat{b}_\alpha {}^1\tilde{V}''_{\mathbf{k}\alpha}(s) = \frac{s + \bar{\Omega}}{s + 2\Omega} {}^0\hat{b}_\alpha \tilde{U}_{\mathbf{k}\alpha}(s). \quad (3.59)$$

Thus, the growing velocity modes  ${}^1\tilde{\mathbf{V}}'_\mathbf{k}$  are perpendicular to both vectors  $\hat{\mathbf{k}}$  and  ${}^0\hat{\mathbf{b}}$  (of course, all modes must be perpendicular to  $\hat{\mathbf{k}}$  because of the fluid incompressibility condition). At the same time, the other, non-growing modes,  ${}^1\tilde{\mathbf{V}}''_\mathbf{k}$ , have nonzero components along the initial magnetic field unit vector  ${}^0\hat{\mathbf{b}}$ . In addition, equation (3.43) means that  ${}^1\tilde{\mathbf{V}}'_\mathbf{k}$  and  ${}^1\tilde{\mathbf{V}}''_\mathbf{k}$  are perpendicular on average.

These results can be understood by an independent physical argument that appeals to the Braginskii viscous dissipation process. The viscous dissipation into heat

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<sup>5</sup> This can also be seen because of the following simple arguments. The poles of function  $\langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(s) {}^1\tilde{V}'_{\mathbf{k}'\beta}(s') \rangle$  are  $s = 0$ ,  $s' = 0$ ,  $s = -i\omega$  and  $s' = i\omega$ , see equation (3.44). All these poles have nonnegative real parts, and there is no viscous damping of  $\langle |{}^1\tilde{\mathbf{V}}'_\mathbf{k}(t)|^2 \rangle$ . On the other hand, the poles of function  $\langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(s) {}^1\tilde{V}''_{\mathbf{k}'\beta}(s') \rangle$  are  $s = -2\Omega$ ,  $s' = -2\Omega$ ,  $s = -i\omega$  and  $s' = i\omega$ , see equation (3.45). Two poles are negative, and  $\langle |{}^1\tilde{\mathbf{V}}''_\mathbf{k}(t)|^2 \rangle$  is viscously damped with a rate  $\sim \Omega$ .

is [7]

$$Q_{\text{vis}} = \pi_{\alpha\beta} V_{\alpha,\beta} = -3\nu \left( {}^1V_{\alpha,\beta} {}^0\hat{b}_\alpha {}^0\hat{b}_\beta \right)^2 = -3\nu \left( {}^1V''_{\alpha,\beta} {}^0\hat{b}_\alpha {}^0\hat{b}_\beta \right)^2, \quad (3.60)$$

where we keep only the first order terms in the parentheses, and make use of equation (3.58). Using this formula, the inverse Laplace transform of equation (3.45) [which is the second term in the square brackets in equation (3.51)] and definition (3.38), we obtain the ensemble averaged dissipation

$$\begin{aligned} \langle Q_{\text{vis}} \rangle &= -3\nu \langle {}^1V_{\alpha,\beta} {}^1V_{\gamma,\tau} \rangle {}^0b_{\alpha\beta\gamma\tau} = 3\nu \sum_{\mathbf{k}, \mathbf{k}'} k_\beta k'_\tau \left\langle {}^1\tilde{V}''_{\mathbf{k}\alpha} {}^1\tilde{V}''_{\mathbf{k}'\gamma} \right\rangle {}^0b_{\alpha\beta\gamma\tau} e^{i(\mathbf{k}+\mathbf{k}')\mathbf{r}} \\ &= -3\nu \sum_{\mathbf{k}} k^2 \mu^2 {}^0\hat{b}_\alpha \langle {}^1\tilde{V}''_{\mathbf{k}\alpha} {}^1\tilde{V}''_{-\mathbf{k}\gamma} \rangle {}^0\hat{b}_\gamma \\ &= -\sum_{\mathbf{k}} 2\Omega \frac{1}{2\pi} \int_{-\infty}^{\infty} J_{\omega k} H_L(t; \omega, \Omega) H_L(t'; -\omega, \Omega) d\omega. \end{aligned} \quad (3.61)$$

We see that first, only  ${}^1\tilde{\mathbf{V}}''_{\mathbf{k}}$  velocity modes are dissipated by the Braginskii viscous forces, and second, the viscous dissipation is proportional to  $2\Omega = 3\nu k^2 \mu^2 (1 - \mu^2)$ . As a result, we can conclude that the growth of the first order velocity happens because the  ${}^1\tilde{\mathbf{V}}'_{\mathbf{k}}$  velocity modes, which are perpendicular to  ${}^0\hat{\mathbf{b}}$ , are not damped by the Braginskii viscous dissipation process. Of course, in the degenerate cases, when  $\hat{\mathbf{k}} \perp {}^0\hat{\mathbf{b}}$  (i. e.  $\mu^2 = 0$ ) or  $\hat{\mathbf{k}} \parallel {}^0\hat{\mathbf{b}}$  (i. e.  $1 - \mu^2 = 0$ ), both velocity modes  ${}^1\tilde{\mathbf{V}}'_{\mathbf{k}}$  and  ${}^1\tilde{\mathbf{V}}''_{\mathbf{k}}$  are undamped.

It is clear that the growing velocity modes can not grow unrestrictedly. What are the mechanisms which stop this growth? To answer this question, let us again note that the only growing velocity modes are those which are perpendicular to  $\hat{\mathbf{b}}$ . All other modes are damped by the Braginskii viscosity. For example, in our quasi-linear expansion the zero order magnetic field unit vector  ${}^0\hat{b}$  is assumed to be stationary,

see equation (3.6), and therefore, the first order velocity mode  ${}^1\tilde{\mathbf{V}}_{\mathbf{k}}'(t)$ , which is perpendicular to  ${}^0\hat{\mathbf{b}}$ , can unrestrictedly grow in time. However, in reality, any velocity vector continuously rotates relative to the magnetic field unit vector. As a result, the growing modes do not stay perpendicular to  $\hat{\mathbf{b}}$ , and therefore, they do not grow in time forever. We can understand this rotation from different standpoints. First, in the laboratory reference frame the velocity vector and the magnetic field unit vector both rotate, but they must rotate differently because different “forces” are acting on them [in other words, because they satisfy different differential equations]. Second, we can go to a reference frame that has the origin at a given point of space and rotates together with the field unit vector at this point. In this rotating reference frame there exist Coriolis forces which act on velocity vectors and force them to rotate relative to the non-rotating field vector. These Coriolis forces are caused by the non-linear coupling of velocity modes via the inertial term  $(\mathbf{V}\nabla)\mathbf{V}$  of the MHD equation (2.29). Consequently, the rotation is produced by the non-linear coupling of velocity modes.

Thus, a growing velocity mode, which is perpendicular to the magnetic field unit vector, rotates out of its initial direction. After this, the velocity mode is viscously damped, and stops growing. We call this process the “effective rotational damping” of velocity modes. Of course, the effective rotational damping operates only on the viscous scale, on which the Braginskii viscous dissipation is significant. On larger scales the viscous dissipation is small and the rotation of velocities does not make any difference. Now, let us estimate the effective damping rate of the growing velocity modes, which is associated with the rotational damping process. First, the square of the angular velocity of the rotation can be estimated as

$$\omega_{\text{rot}}^2 \sim \frac{1}{3} \frac{1}{\tau_\nu^2}, \quad (3.62)$$

where  $\tau_\nu$  is the velocity decorrelation time on the viscous scale. The factor  $1/3$  in this equation comes from the fact that only one of the three angular velocity components contributes to the deviation of the velocity  $\mathbf{V}$  from its original direction perpendicular to the magnetic field unit vector  $\hat{\mathbf{b}}$ . Namely, only the component along the vector product  $\hat{\mathbf{b}} \times \mathbf{V}$  contributes. Second, the typical Braginskii viscous damping rate is about  $1/\tau_\nu \sim \nu_{\text{eff}} k_\nu^2$  because of the definition of the viscous wave number  $k_\nu$ , as the wave number at which the viscous dissipation becomes important. Thus, the typical angular velocity of the rotation of the velocity modes is less than the typical viscous damping rate. Therefore, we assume that after a time interval  $\Delta t$  the growing velocity mode rotates by an angle  $\Delta\phi$  relative to its original direction perpendicular to  $\hat{\mathbf{b}}$ , only the projection of the velocity mode on an undamped direction (which is in the plane perpendicular to  $\hat{\mathbf{b}}$ ) survives, and all other components of the velocity mode are immediately viscously dissipated. As a result, the effective rotational damping rate  $\Omega_{\text{rd}}$  can be estimated as

$$\begin{aligned} \frac{dV}{dt} &\sim \frac{\Delta V}{\Delta t} \sim \frac{V(\cos \Delta\phi - 1)}{\Delta t} \sim -\frac{V}{2\Delta t} \Delta\phi^2 \sim -\frac{V}{2\Delta t} (\omega_{\text{rot}}\tau_\nu)^2 \frac{\Delta t}{\tau_\nu} \\ &\sim -\frac{1}{6\tau_\nu} V \sim \frac{1}{6}\nu_{\text{eff}} k_\nu^2 V^2 \sim \frac{1}{30}\nu k_\nu^2 V^2 = -\Omega_{\text{rd}} V^2, \end{aligned} \quad (3.63)$$

$$\Omega_{\text{rd}} = \frac{1}{6}\nu_{\text{eff}} k_\nu^2 = \frac{1}{30}\nu k_\nu^2. \quad (3.64)$$

To obtain the last result in the first line of equation (3.63), we use the random-walk approximation for the estimate of  $\Delta\phi^2$ .<sup>6</sup> We also use formula (3.62) for  $\omega_{\text{rot}}$ .

As we said, the effective rotational damping is associated with non-linear coupling of velocity modes. On one hand, this non-linearity is hard to deal with directly. On

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<sup>6</sup> Basically, here we assume that  $\omega_{\text{rot}}^2 \tau_\nu^2 \sim 1/3 \ll 1$ . In this case we can choose such time interval  $\Delta t \gg \tau_\nu$  that  $\Delta\phi \sim \omega_{\text{rot}} \Delta t \ll 1$ . As a result, on one hand, we can expand the cosine in equation (3.63). On the other hand, the turbulent eddies on the viscous scale are correlated on time intervals  $\sim \tau_\nu \ll \Delta t$ , and we can use the random-walk approximation to estimate  $\Delta\phi^2$ .

the other hand, the rotational damping is physical and is very important, it has to be included into the theory in order to restrict the growth of the velocity modes undamped by the Braginskii viscous forces. Therefore, we are going to incorporate the rotational damping into our equations in a simple way as follows. First, we go back to the original equations for velocities, equations (2.29) and (2.34). We recall that the total velocity  $\mathbf{V}$  is the sum of the Kolmogorov hydrodynamic turbulent velocity  $\mathbf{U}$ , which is assumed to be given, and of the back-reaction velocity  $\mathbf{v}$ , which we obtain by solving equation (2.34). The Kolmogorov turbulence is assumed to be stationary (and homogeneous). Thus, the growth of the undamped modes of the total velocity  $\mathbf{V} = \mathbf{U} + \mathbf{v}$  happens because the corresponding modes of the back-reaction velocity  $\mathbf{v}$  are undamped and grow <sup>7</sup>. The rotational damping restricts the growth of these back-reaction velocity modes on the viscous scale. As for the scales larger than the viscous scales, the turbulence is Kolmogorov on these large scales,  $\mathbf{V} = \mathbf{U}$ , and the back reaction velocities are zero [see the discussion in Section 2.2, also note that the driving force (3.28), which is  $\propto k^2$ , becomes small on large scales]. As a result, on all scales we add the rotational non-linear damping into our equations for the velocities  $\mathbf{v}$ . We do this by replacing the time derivative  $\partial/\partial t$  by  $\partial/\partial t + \Omega_{\text{rd}}$  in equation (2.34):

$$\begin{aligned}
(\partial_t + \Omega_{\text{rd}})v_\alpha &= -P_{,\alpha} + 3\nu(b_{\alpha\beta\mu\nu}v_{\mu,\nu})_{,\beta} + 3\nu(b_{\alpha\beta\mu\nu}U_{\mu,\nu})_{,\beta} - \frac{1}{5}\nu\Delta U_\alpha \\
&\quad - (v_\alpha U_\beta + U_\alpha v_\beta + v_\alpha v_\beta)_{,\beta} ,
\end{aligned} \tag{3.65}$$

$$v_{\alpha,\alpha} = 0. \tag{3.66}$$

Once again, the purpose of this replacement is to incorporate the rotational non-linear damping, which stops the unrestricted growth of the velocity modes undamped by

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<sup>7</sup> One can obtain the Fourier-Laplace coefficient of the first order back-reaction velocity,  ${}^1\tilde{\mathbf{v}}_{\mathbf{k}}$ , from equation (3.30), and then derive formulas for the growth of  $\langle {}^1\mathbf{v}^2(t) \rangle$ , which are very similar to equations (3.53)–(3.55).

the Braginskii viscosity, into our equations in a simple way, by including the linear damping term  $\Omega_{\text{rd}}\mathbf{v}_\alpha$ . This is our *second working hypothesis*. A possible check of it can be intensive MHD numerical simulations, including the Braginskii viscosity. However, such simulations are rather complicated <sup>8</sup>, and they are beyond the scope of this thesis. Note, that in equation (3.65), the effective rotational damping term  $\Omega_{\text{rd}}\mathbf{v}_\alpha$  is isotropic, and therefore, it damps not only the growing modes of  $\mathbf{v}$  (which are undamped by the Braginskii viscosity) but all  $\mathbf{v}$  modes. This is not a serious problem though, because the rotational damping is smaller than the Braginskii damping by a factor  $\sim 1/6$  [see equation (3.64)], and our results should be valid within a factor of order unity.

### 3.4 The Time Fourier Transform Solution for the Velocities

In this section we carry out calculations similar to those of Section 3.2, again assuming our first working hypothesis that the initial field can be taken to be straight,  ${}^0b_{\alpha\beta} = \text{const.}$  We also again use the quasi-linear expansion (3.1)–(3.4). The main difference is that now, instead of using equation (2.34) for the back-reaction velocity, we use equation (3.65) with the effective rotational damping (3.64) in it. In the end of this section we finally obtain convenient formulas for the statistics of turbulent velocities in a strongly magnetized plasma.

The first and the second order equations (3.13) and (3.20) for the back-reaction

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<sup>8</sup> In particular because the Braginskii forces are anisotropic and there are undamped velocity modes, it is not possible to drop the inertial terms in the MHD equations (i. e. not possible to assume the viscosity dominated regime). For example, if we drop the inertial term  $s\delta_{\alpha\beta}$  in equation (3.31), then the matrix  ${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta}$  will not have an inverse matrix.

velocity become

$$\begin{aligned}
(\partial_t + \Omega_{\text{rd}}) {}^1v_\alpha &= - {}^1P_{,\alpha} + 3\nu({}^0b_{\alpha\beta\mu\nu} {}^1v_{\mu,\nu})_{,\beta} + 3\nu({}^0b_{\alpha\beta\mu\nu} U_{\mu,\nu})_{,\beta} \\
&\quad - \frac{1}{5}\nu\Delta U_\alpha, \tag{3.67}
\end{aligned}$$

$$\begin{aligned}
(\partial_t + \Omega_{\text{rd}}) {}^2v_\alpha &= - {}^2P_{,\alpha} + \left( 3\nu {}^1b_{\alpha\beta\mu\nu} {}^1v_{\mu,\nu} + 3\nu {}^0b_{\alpha\beta\mu\nu} {}^2v_{\mu,\nu} + 3\nu {}^1b_{\alpha\beta\mu\nu} U_{\mu,\nu} \right. \\
&\quad \left. - {}^1v_\alpha U_\beta - U_\alpha {}^1v_\beta - {}^1v_\alpha {}^1v_\beta \right)_{,\beta}, \tag{3.68}
\end{aligned}$$

and they now include the rotational damping terms in them. All other first and second order equations (3.11)–(3.12), (3.14)–(3.19) and (3.21)–(3.22) stay unchanged. In Section 3.2, to solve the first and the second order equations, we use the Fourier and the Laplace transformations in space and in time respectively. In this section it is convenient to use the Fourier transformations both in space and in time<sup>9</sup>. We use the discrete Fourier transformation  $\mathbf{r} \rightarrow \mathbf{k}$  in space and the continuous Fourier transformation  $t \rightarrow \omega$  in time (see Appendix A). Comparing equations (3.13) and (3.67), and equations (A.9) and (A.12), we easily see that the double Fourier transformation (in space and in time) of equation (3.67), which is the first order equation for the back-reaction velocity Fourier coefficient  ${}^1\tilde{u}_{\mathbf{k},\alpha}(\omega)$ , coincides with equation (3.23) where the Laplace variable  $s$  is replaced by  $-i\omega + \Omega_{\text{rd}}$ . As a result, there is no need to rederive all equations of Section 3.2. In most cases we can obtain the new formulas just by making the replacement  $s \rightarrow -i\omega + \Omega_{\text{rd}}$  in the corresponding formulas of Section 3.2. However, let us be more general, and in the calculations below assume that the effective rotational damping rate  $\Omega_{\text{rd}}$  is a function of  $k = |\mathbf{k}|$  and of  $(\hat{\mathbf{b}} \cdot \hat{\mathbf{k}})^2$ ,<sup>10</sup>

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<sup>9</sup> Using the Fourier transformation in time is actually better than using the Laplace transformation. The reason is that in the case of the Laplace transformation we have to assume the initial condition on the back-reaction velocity (in Section 3.2 we assume  ${}^1\boldsymbol{v}|_{t=0}$ ), while the Fourier transformation “does not care” about the initial condition. As a result, we do not have to worry about any velocity transients, which are eventually damped away by the Braginskii viscosity and by the rotational damping.

<sup>10</sup> We choose  $\Omega_{\text{rd}}$  to depend on the square of  $\hat{\mathbf{b}} \cdot \hat{\mathbf{k}}$  because the Braginskii viscosity is invariant

e. g.  $\Omega_{\text{rd}} = \Omega_{\text{rd}}(k, \mu^2)$  up to the zero order, where  $\mu$  is given by equation (3.32). We can always substitute our simple estimate (3.64) for  $\Omega_{\text{rd}}$  into our final formulas.

Equations (3.34)–(3.36) for the Fourier coefficient of the first order velocity now become

$${}^1\tilde{V}_{\mathbf{k}\alpha}(\omega) = {}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) + {}^1\tilde{V}''_{\mathbf{k}\alpha}(\omega), \quad (3.69)$$

$${}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) = \frac{-i\omega + \bar{\Omega} + \Omega_{\text{rd}}}{-i\omega + \Omega_{\text{rd}}} \left[ \delta_{\alpha\beta} - \frac{\delta_{\alpha\gamma}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\beta}}{1 - \mu^2} \right] \tilde{U}_{\mathbf{k}\beta}(\omega), \quad (3.70)$$

$${}^1\tilde{V}''_{\mathbf{k}\alpha}(\omega) = \frac{-i\omega + \bar{\Omega} + \Omega_{\text{rd}}}{-i\omega + 2\Omega + \Omega_{\text{rd}}} \frac{\delta_{\alpha\gamma}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\beta}}{1 - \mu^2} \tilde{U}_{\mathbf{k}\beta}(\omega). \quad (3.71)$$

In these formulas the effective rotational damping rate,  $\Omega_{\text{rd}}$ , is added to the viscous damping frequencies (3.37) and (3.38). Thus, the overall damping is produced by both the rotational damping rate and the viscous damping (of course, there is only one physical dissipation process present in the fluid — the viscous dissipation). Now, we use formulas (2.4), (2.5) to obtain averages similar to those given by equations (3.41)–(3.45). We have

$$\langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) \rangle = 0, \quad (3.72)$$

$$\langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(\omega) \rangle = 0, \quad (3.73)$$

$$\langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}''_{\mathbf{k}'\beta}(\omega') \rangle = 0, \quad (3.74)$$

$$\begin{aligned} \langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}'_{\mathbf{k}'\beta}(\omega') \rangle &= J_{\omega k} \delta_{\mathbf{k}', -\mathbf{k}} \delta(\omega' + \omega) \frac{\omega^2 + (\bar{\Omega} + \Omega_{\text{rd}})^2}{\omega^2 + \Omega_{\text{rd}}^2} \\ &\times \left[ \delta_{\alpha\beta}^{\perp} - \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right], \end{aligned} \quad (3.75)$$

$$\langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}''_{\mathbf{k}'\beta}(\omega') \rangle = J_{\omega k} \delta_{\mathbf{k}', -\mathbf{k}} \delta(\omega' + \omega) \frac{\omega^2 + (\bar{\Omega} + \Omega_{\text{rd}})^2}{\omega^2 + (2\Omega + \Omega_{\text{rd}})^2}$$

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under reflection  $\hat{\mathbf{b}} \rightarrow -\hat{\mathbf{b}}$ .

$$\times \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2}. \quad (3.76)$$

Using these equations and equation (3.69), we obtain equations, which are similar to equations (3.46) and (3.47) of Section 3.2,

$$\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(\omega) \rangle = 0, \quad (3.77)$$

$$\begin{aligned} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}_{\mathbf{k}'\beta}(\omega') \rangle &= \langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}'_{\mathbf{k}'\beta}(\omega') \rangle + \langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}''_{\mathbf{k}'\beta}(\omega') \rangle \\ &= J_{\omega k} \delta_{\mathbf{k}', -\mathbf{k}} \delta(\omega' + \omega) \\ &\times \left[ \tilde{H}_F(\omega; \bar{\Omega} + \Omega_{\text{rd}}, \Omega_{\text{rd}}) \left( \delta_{\alpha\beta}^{\perp} - \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right) \right. \\ &\quad \left. + \tilde{H}_F(\omega; \bar{\Omega} + \Omega_{\text{rd}}, 2\Omega + \Omega_{\text{rd}}) \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right], \end{aligned} \quad (3.78)$$

where function  $\tilde{H}_F(\omega; q_1, q_2)$  is

$$\tilde{H}_F(\omega; q_1, q_2) = \frac{\omega^2 + q_1^2}{\omega^2 + q_2^2} = 1 + \frac{q_1^2 - q_2^2}{\omega^2 + q_2^2}. \quad (3.79)$$

Next, we apply the inverse Fourier transformations  $\omega \rightarrow t$  and  $\omega' \rightarrow t'$  to equations (3.77) and (3.78). We have

$$\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) \rangle = 0, \quad (3.80)$$

$$\begin{aligned} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{\mathbf{k}'\beta}(t') \rangle &= \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{-\mathbf{k}'\beta}^*(t') \rangle \\ &= \langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(t) {}^1\tilde{V}'_{\mathbf{k}'\beta}(t') \rangle + \langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(t) {}^1\tilde{V}''_{\mathbf{k}'\beta}(t') \rangle \\ &= \delta_{\mathbf{k}', -\mathbf{k}} \left[ H_F(t - t'; \bar{\Omega} + \Omega_{\text{rd}}, \Omega_{\text{rd}}) \left( \delta_{\alpha\beta}^{\perp} - \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right) \right. \\ &\quad \left. + H_F(t - t'; \bar{\Omega} + \Omega_{\text{rd}}, 2\Omega + \Omega_{\text{rd}}) \frac{\delta_{\alpha\gamma}^{\perp} \delta_{\beta\tau}^{\perp} {}^0\hat{b}_{\gamma} {}^0\hat{b}_{\tau}}{1 - \mu^2} \right], \end{aligned} \quad (3.81)$$

where function

$$\begin{aligned}
H_F(t - t'; q_1, q_2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} J_{\omega k} \tilde{H}_F(\omega; q_1, q_2) e^{-i\omega(t-t')} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} J_{\omega k} \tilde{H}_F(\omega; q_1, q_2) \cos[\omega(t - t')] d\omega \quad (3.82)
\end{aligned}$$

depends only on the absolute value of the time difference  $t - t'$  because  $J_{\omega k}$  and  $\tilde{H}_F(\omega; q_1, q_2)$  both are even functions of  $\omega$ .

Equations (3.80)–(3.81) are similar to equations (3.50)–(3.51) of Section 3.2,<sup>11</sup> and they give the statistics of the first order turbulent velocities, which we will use in the next chapter. As, for the second order turbulent velocity, equation (3.52) stay the same,

$$\langle {}^2\tilde{V}_{\mathbf{k}\alpha}(t) \rangle = 0, \quad (3.83)$$

because the proof of this formula, given in Appendix B, does not involve any transformation in time.

In the end of this section let us substitute equation (2.10) for  $J_{\omega k}$  and equation (3.79) for  $\tilde{H}_F(\omega; q_1, q_2)$  into equation (3.82), and obtain the following formulas

$$\begin{aligned}
H_F(0; q_1, q_2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J_{0k}}{1 + \tau^2 \omega^2} \left[ 1 + \frac{q_1^2 - q_2^2}{\omega^2 + q_2^2} \right] d\omega \\
&= \frac{J_{0k}}{2\tau} + \frac{J_{0k}}{2} \frac{q_1^2 - q_2^2}{q_2(1 + \tau q_2)}, \quad (3.84)
\end{aligned}$$

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<sup>11</sup> It is interesting, that we can obtain equations (3.80)–(3.82) from equations (3.49)–(3.51) by first, making replacements  $\bar{\Omega} \rightarrow \bar{\Omega} + \Omega_{\text{rd}}$ ,  $2q_2 \rightarrow 2q_2 + \Omega_{\text{rd}}$  in equation (3.49), and second, by taking a limit  $t \rightarrow \infty$  in it. This is not surprising, because as time goes on, the velocity transients, which “remember” the initial conditions, are damped away by the Braginskii viscosity and by the rotational damping.

$$\begin{aligned}
\int_0^t H_F(t-t') dt' &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{J_{0k}}{1+\tau^2\omega^2} \left[ 1 + \frac{q_1^2 - q_2^2}{\omega^2 + q_2^2} \right] \frac{\sin \omega t}{\omega} d\omega \\
&= \frac{J_{0k}}{2} (1 - e^{-t/\tau}) + \frac{J_{0k}}{2} \frac{q_1^2 - q_2^2}{1 - \tau^2 q_2^2} \left[ \frac{1 - e^{-q_2 t}}{q_2^2} - \tau^2 (1 - e^{-t/\tau}) \right], \\
&\rightarrow \frac{J_{0k}}{2} + \frac{J_{0k}}{2} \frac{q_1^2 - q_2^2}{q_2^2} = \frac{J_{0k}}{2} \frac{q_1^2}{q_2^2}, \tag{3.85}
\end{aligned}$$

$$\begin{aligned}
\int_0^t dt' \int_0^{t'} H_F(t' - t'') dt'' &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{J_{0k}}{1+\tau^2\omega^2} \left[ 1 + \frac{q_1^2 - q_2^2}{\omega^2 + q_2^2} \right] \frac{\sin^2(\omega t/2)}{\omega^2} d\omega \\
&= \frac{J_{0k}}{2} [t - \tau(1 - e^{-t/\tau})] \\
&\quad + \frac{J_{0k}}{2} \frac{q_1^2 - q_2^2}{1 - \tau^2 q_2^2} \left[ \frac{q_2 t - 1 + e^{-q_2 t}}{q_2^3} - \tau^2 t + \tau^3 (1 - e^{-t/\tau}) \right] \\
&\rightarrow \frac{J_{0k}}{2} t + \frac{J_{0k}}{2} \frac{q_1^2 - q_2^2}{q_2^2} t = \frac{J_{0k}}{2} \frac{q_1^2}{q_2^2} t, \tag{3.86}
\end{aligned}$$

$$\int_0^t dt' \int_0^{t'} H_F(t' - t'') dt'' = 2 \int_0^t dt' \int_0^{t'} H_F(t' - t'') dt'' \rightarrow J_{0k} \frac{q_1^2}{q_2^2} t, \tag{3.87}$$

which we will use below. The integrals over  $\omega$  can be done by closing the integration contours in the complex plane and by evaluating the residues. The final answers in formulas (3.85)–(3.87), which are given after the right arrows  $\rightarrow$ , give the results in the limit  $t \gg \tau$ ,  $t \gg q_2^{-1}$ .

Using equations (3.81) and (3.84), we easily obtain

$$\begin{aligned}
\langle |\mathbf{1}\tilde{\mathbf{V}}_{\mathbf{k}}(t)|^2 \rangle &= H_F(0; \bar{\Omega} + \Omega_{\text{rd}}, \Omega_{\text{rd}}) + H_F(0; \bar{\Omega} + \Omega_{\text{rd}}, 2\Omega + \Omega_{\text{rd}}) \\
&= \frac{J_{0k}}{\tau} + \frac{J_{0k}}{2} \frac{(\bar{\Omega} + \Omega_{\text{rd}})^2 - \Omega_{\text{rd}}^2}{\Omega_{\text{rd}}(1 + \tau\Omega_{\text{rd}})} \\
&\quad + \frac{J_{0k}}{2} \frac{(\bar{\Omega} + \Omega_{\text{rd}})^2 - (2\Omega + \Omega_{\text{rd}})^2}{(2\Omega + \Omega_{\text{rd}})[1 + \tau(2\Omega + \Omega_{\text{rd}})]}. \tag{3.88}
\end{aligned}$$

Substituting this equation into equation (3.53), we see that the ensemble average of  $\mathbf{1}\mathbf{V}$  squared is finite, but it would be infinite if the rotational damping were absent (i. e. if  $\Omega_{\text{rd}} = 0$ ).

# Chapter 4

## Energy Spectrum of Random Magnetic Fields

In this chapter we use equations (3.80)–(3.83) for the statistics of turbulent velocities to find the evolution of the magnetic field energy and of the magnetic energy spectrum in the magnetized turbulent dynamo theory. The principal results of this chapter (and of the thesis) are given by the following equations. The growth of the total magnetic energy is given by equations (4.16)–(4.20). The evolution of the magnetic energy spectrum is given by equations (4.56)–(4.63). The evolution of the magnetic energy spectrum on small (subviscous) scales is given by equations (4.88)–(4.99).

### 4.1 The Growth of the Total Magnetic Energy

We start with calculation of the magnetic field energy growth because it is of the greatest interest. The volume averaged and ensemble averaged <sup>1</sup> magnetic energy per

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<sup>1</sup> Ensemble averaged over the ensemble of turbulent forces.

unit mass is

$$\mathcal{E} \stackrel{\text{def}}{=} \frac{1}{L^3} \int_{-L/2}^{L/2} \frac{\langle B^2 \rangle}{8\pi\rho} d^3\mathbf{r} = \frac{1}{8\pi\rho} \langle \widetilde{B^2}_{\mathbf{k}=0} \rangle, \quad (4.1)$$

where  $\rho$  is the plasma density, and  $\widetilde{B^2}_{\mathbf{k}=0}$  is the  $\mathbf{k} = 0$  Fourier coefficient of the magnetic field strength squared,  $B^2$ , see equation (A.5) in Appendix A. To find  $\mathcal{E}(t)$ , it is convenient to introduce the following symmetric tensor

$$B_{\alpha\beta} \stackrel{\text{def}}{=} B_\alpha B_\beta = B^2 b_{\alpha\beta}, \quad (4.2)$$

where  $b_{\alpha\beta}$  is given by equation (2.38). The differential equation for  $B_{\alpha\beta}$  can easily be derived from equation (2.39),

$$\partial_t B_{\alpha\beta} = B_\alpha \partial_t B_\beta + B_\beta \partial_t B_\alpha = V_{\alpha,\gamma} B_{\beta\gamma} + V_{\beta,\gamma} B_{\alpha\gamma} - V_\gamma B_{\alpha\beta,\gamma}. \quad (4.3)$$

Now, we solve this equation and equation (2.40) by making use of the quasi-linear expansion procedure, described in Section 3.1. First, we write the expansion for  $B^2$  up to the second order, and for  $B_{\alpha\beta}$  up to the first order <sup>2</sup>,

$$B^2(t) = {}^0B^2 + {}^1B^2(t) + {}^2B^2(t), \quad (4.4)$$

$$B_{\alpha\beta}(t) = {}^0B_{\alpha\beta} + {}^1B_{\alpha\beta}(t). \quad (4.5)$$

Here, at zero time  $B^2(0) = {}^0B^2$ ,  ${}^1B^2(0) = 0$ ,  ${}^2B^2(0) = 0$ ,  $B_{\alpha\beta}(0) = {}^0B_{\alpha\beta}$  and  ${}^1B_{\alpha\beta}(0) = 0$ . Second, we substitute these expansion formulas into equations (2.40) and (4.3).

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<sup>2</sup> Note, that below the magnetic field strength squared  $B^2$  is expanded as a whole. Thus, for example, the first order quantity  ${}^1B^2$  in equation (4.4) is not equal to  ${}^1\mathbf{B} \cdot {}^1\mathbf{B}$ , where  ${}^1\mathbf{B}$  is the first order expansion term in equation (3.1), the later is of the second order. Of course,  ${}^1B^2$  is equal to  $2({}^0\mathbf{B} \cdot {}^1\mathbf{B})$ .

We find that the first order equations are

$$\partial_t {}^1B^2 = 2 {}^1V_{\alpha,\beta} {}^0B_{\alpha\beta} - {}^1V_{\beta} {}^0B^2_{,\beta}, \quad (4.6)$$

$$\partial_t {}^1B_{\alpha\beta} = {}^1V_{\alpha,\gamma} {}^0B_{\beta\gamma} + {}^1V_{\beta,\gamma} {}^0B_{\alpha\gamma} - ({}^1V_{\gamma} {}^0B_{\alpha\beta})_{,\gamma}, \quad (4.7)$$

and the second order equation for  ${}^2B^2(t)$  is

$$\partial_t {}^2B^2 = 2 {}^1V_{\alpha,\beta} {}^1B_{\alpha\beta} + 2 {}^2V_{\alpha,\beta} {}^0B_{\alpha\beta} - ({}^1V_{\beta} {}^1B^2)_{,\beta} - ({}^2V_{\beta} {}^0B^2)_{,\beta}. \quad (4.8)$$

Here, in the last two equations we transform the last (convective) terms by making use of the plasma incompressibility condition,  $V_{\alpha,\alpha} = 0$ .

Next, we first integrate first order equation (4.6) in time (with the zero initial conditions) and ensemble average the result. Using equation (3.80), we obviously obtain

$$\langle {}^1B^2(t) \rangle = 0. \quad (4.9)$$

Second, we integrate equation (4.7) in time (with the zero initial conditions), and then Fourier transform the result in space,  $\mathbf{r} \rightarrow \mathbf{k}$ . We have

$$\begin{aligned} {}^1\tilde{B}_{\mathbf{k}\alpha\beta}(t) &= i \int_0^t \left[ k'_{\gamma} \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} {}^1\tilde{V}_{\mathbf{k}'\alpha}(t') {}^0\tilde{B}_{\mathbf{k}''\beta\gamma} + k'_{\gamma} \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} {}^1\tilde{V}_{\mathbf{k}'\beta}(t') {}^0\tilde{B}_{\mathbf{k}''\alpha\gamma} \right. \\ &\quad \left. - k_{\gamma} \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} {}^1\tilde{V}_{\mathbf{k}'\gamma}(t') {}^0\tilde{B}_{\mathbf{k}''\alpha\beta} \right] dt' \\ &= i \left[ k'_{\gamma} (\delta_{\alpha\tau} {}^0b_{\beta\gamma} + \delta_{\beta\tau} {}^0b_{\alpha\gamma}) - k_{\tau} {}^0b_{\alpha\beta} \right] \int_0^t \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} {}^1\tilde{V}_{\mathbf{k}'\tau}(t') {}^0\tilde{B}^2_{\mathbf{k}''} dt'. \end{aligned} \quad (4.10)$$

Here, to obtain the last line of this equation, we use equation (4.2) and we assume

that the magnetic field lines can be considered as initially straight,  ${}^0b_{\alpha\beta} = \text{const}$ , (this is our first working hypothesis, see Section 3.2). Third, we integrate the second order equation (4.8) in time (with the zero initial conditions) and ensemble average the result. Using equation (3.83), we obtain

$$\langle {}^2B^2(t) \rangle = \int_0^t \left[ 2 \langle {}^1V_{\alpha,\beta}(t') {}^1B_{\alpha\beta}(t') \rangle - \langle {}^1V_{\beta}(t') {}^1B^2(t') \rangle_{,\beta} \right] dt'. \quad (4.11)$$

Fourth, we Fourier transform this result in space,  $\mathbf{r} \rightarrow \mathbf{k}$ , and set  $\mathbf{k}$  to zero. The second (convective) term in the brackets [...] in equation (4.11) gives zero contribution to the final result <sup>3</sup>, and we have

$$\langle {}^2\widetilde{B}^2_{\mathbf{k}=0}(t) \rangle = 2i \int_0^t \sum_{\mathbf{k}} k_{\beta} \langle {}^1\widetilde{V}_{\mathbf{k}\alpha}(t') {}^1\widetilde{B}_{-\mathbf{k}\alpha\beta}(t') \rangle dt'. \quad (4.12)$$

Fifth, we substitute equation (4.10) into this last equation, and use equation (3.81). We obtain

$$\begin{aligned} \langle {}^2\widetilde{B}^2_{\mathbf{k}=0}(t) \rangle &= 2 {}^0\widetilde{B}^2_{\mathbf{k}=0} \int_0^t dt' \int_0^{t'} \sum_{\mathbf{k}} \mu^2 k^2 \langle {}^1\widetilde{V}_{\mathbf{k}\alpha}(t') {}^1\widetilde{V}_{-\mathbf{k}\alpha}(t'') \rangle dt'' \\ &= 2 {}^0\widetilde{B}^2_{\mathbf{k}=0} \sum_{\mathbf{k}} \mu^2 k^2 \int_0^t dt' \int_0^{t'} \left[ H_F(t' - t''; \bar{\Omega} + \Omega_{\text{rd}}, \Omega_{\text{rd}}) \right. \\ &\quad \left. + H_F(t' - t''; \bar{\Omega} + \Omega_{\text{rd}}, 2\Omega + \Omega_{\text{rd}}) \right] dt'' \\ &= 2\gamma t {}^0\widetilde{B}^2_{\mathbf{k}=0}, \end{aligned} \quad (4.13)$$

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<sup>3</sup> This is not surprising because the convective term can only redistribute magnetic energy in space, but can not change the volume averaged energy.

where

$$\gamma = \frac{1}{2} \sum_{\mathbf{k}} k^2 J_{0k} \left(1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}}\right)^2 \mu^2 \left[1 + \left(1 + \frac{2\Omega}{\Omega_{\text{rd}}}\right)^{-2}\right]. \quad (4.14)$$

Here, we also use equation (3.86) in the limit  $t \gg \tau$ .

Now, we can finally obtain the differential equation for the averaged magnetic energy  $\mathcal{E}$ . Following Kulsrud and Anderson [27], we choose  $t$  small enough for the quasi-linear expansion to be valid, but large enough for the limit  $t \gg \tau$  to be satisfied.

As a result, using equations (4.4), (4.9) and (4.13), we obtain

$$\frac{\partial}{\partial t} \langle \widetilde{B}_{\mathbf{k}=0}^2(t) \rangle = \frac{1}{t} [\langle \widetilde{B}_{\mathbf{k}=0}^2(t) \rangle - \widetilde{B}_{\mathbf{k}=0}^2(0)] = \frac{1}{t} \langle {}^2\widetilde{B}_{\mathbf{k}=0}^2(t) \rangle = 2\gamma {}^0\widetilde{B}_{\mathbf{k}=0}^2, \quad (4.15)$$

and using equation (4.1), we finally obtain

$$\frac{\partial \mathcal{E}}{\partial t} = 2\gamma \mathcal{E}, \quad (4.16)$$

$$\gamma = \pi \left(\frac{L}{2\pi}\right)^3 \int_0^\infty k^4 J_{0k} dk \int_{-1}^1 \mu^2 \left(1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}}\right)^2 \left[1 + \left(1 + \frac{2\Omega}{\Omega_{\text{rd}}}\right)^{-2}\right] d\mu, \quad (4.17)$$

$$\frac{\bar{\Omega}}{\Omega_{\text{rd}}} = 6 \frac{k^2}{k_\nu^2}, \quad (4.18)$$

$$\frac{2\Omega}{\Omega_{\text{rd}}} = 90 \frac{k^2}{k_\nu^2} \mu^2 (1 - \mu^2). \quad (4.19)$$

Here, we replace the summation over  $\mathbf{k}$  in equation (4.14) by integration over  $\mathbf{k}$ , and then use  $d^3\mathbf{k} = 2\pi k^2 dk d\mu$ . We also use equations (3.37), (3.38) and (3.64). Equation (4.16) coincides with formula (2.16) obtained by Kulsrud and Anderson in the case of the kinematic turbulent dynamo. However, the growth rate  $\gamma$ , given by equation (4.17) in the case of the magnetized turbulent dynamo, is different from the growth rate  $\gamma_0$ , given by equation (2.17) in the case of the kinematic dynamo.

Let us now compare these two magnetic energy growth rates. First, if we consider the kinematic turbulent dynamo, then in our formulas we need to take the limit  $\bar{\Omega} \rightarrow 0$ ,  $\Omega \rightarrow 0$ , or alternatively, the limit  $\Omega_{\text{rd}} \rightarrow \infty$ <sup>4</sup>. In this case, after averaging over  $\mu$ , the growth rate (4.17) [see also (4.14)] reduces to the growth rate (2.17), as one can expect. Second, let us use formula (3.64) to make an estimates of the magnetized dynamo growth rate  $\gamma$  and of the ratio  $\gamma/\gamma_o$ . The calculations are given in Appendix C, the result is

$$\gamma \approx 80 \left( \frac{U_0 L}{\nu} \right)^{1/2} \frac{U_0}{L}, \quad (4.20)$$

$$\gamma^{-1} \approx 10^5 \text{ yrs} \left( \frac{\xi}{10} \right)^{1/2} \left( \frac{L}{0.2 \text{ Mpc}} \right)^{3/2}, \quad (4.21)$$

$$\gamma t_{\text{collapse}} \approx 10^4 \left( \frac{\xi}{10} \right)^{-1/2} \left( \frac{M}{10^{12} \text{ M}_\odot} \right)^{-1/2}, \quad (4.22)$$

$$\gamma/\gamma_o \approx 10, \quad (4.23)$$

where to obtain formulas (4.21) and (4.22), we use typical parameters in a protogalaxy given in Table 1.1 (here  $\xi$  is the ratio of the total mass  $M$  to the baryon mass,  $L$  is the protogalaxy size). Thus, our prediction is that the magnetic energy growth rate in the magnetized dynamo theory is up to ten time larger than that in the kinematic dynamo theory. Note, that two different effects contribute to this difference. First, the effective viscosity in the magnetized dynamo theory is smaller than the molecular viscosity,  $\nu_{\text{eff}} = (1/5)\nu < \nu$ , this effect makes the magnetic energy growth rate larger by a factor of square root of five (this factor was included in Kulsrud *et al.* [28]).

The rest of the contribution to the difference between the growth rates comes from

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<sup>4</sup> In this limit the “driving” terms  $3\nu(^0b_{\alpha\beta\mu\nu}U_{\mu,\nu})_{,\beta}$  and  $(1/5)\nu\Delta U_\alpha$  in equation (3.67), which in the Fourier  $\mathbf{k}$ -space result in the two corresponding driving terms proportional to  $\Omega$  and  $\bar{\Omega}$  respectively [e. g. see equation (B.3)], go away. Taking the limit  $\Omega_{\text{rd}} \rightarrow \infty$  is equivalent to an infinitely large damping of the back-reaction velocities. As a result, in both these limits the back-reaction velocities are zero, and the turbulence is Kolmogorov.

the local anisotropy of the turbulent velocities in the strongly magnetized plasma.

Please note, that the number of e-foldings of the magnetic energy during the collapse time,  $\gamma t_{\text{collapse}}$ , given by equation (4.22), does not depend on  $L$ , and therefore, does not depend on the redshift.

## 4.2 The Mode Coupling Equation for the Magnetic Energy Spectrum

In this section we derive the mode coupling equation for the evolution of the magnetic energy spectrum  $M(t, k)$ , given by equation (2.18). Using the quasi-linear expansion formula (3.1) for the magnetic field, we obtain the ensemble averaged square of the magnetic field Fourier coefficient, up to the second order,

$$\begin{aligned} \langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle &= |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 + \left[ \langle {}^1\tilde{B}_{\mathbf{k}\alpha}(t) \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^*(t) + \text{c.c.} \right] \\ &\quad + \langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle + \left[ \langle {}^2\tilde{B}_{\mathbf{k}\alpha}(t) \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^*(t) + \text{c.c.} \right]. \end{aligned} \quad (4.24)$$

We calculate all terms in equation (4.24) separately, our calculations are similar to those of Kulsrud and Anderson [27].

### 4.2.1 The $\langle {}^1\tilde{B}_{\mathbf{k}\alpha}(t) \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.}$ term

We start with the calculation of the  $\langle {}^1\tilde{B}_{\mathbf{k}\alpha}(t) \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.}$  term. We integrate the first order equation (3.11) in time (with the zero initial conditions) and ensemble average the result. Using equation (3.80), we obviously obtain  $\langle {}^1B_{\alpha}(t) \rangle = 0$ . Thus,

$$\langle {}^1\tilde{B}_{\mathbf{k}\alpha}(t) \rangle = 0, \quad (4.25)$$

and the  $\langle {}^1\tilde{B}_{\mathbf{k}\alpha} \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.}$  term on the right-hand-side of equation (4.24) is zero.

### 4.2.2 The $\langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle$ term

Next, we calculate the  $\langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle$  term. First, we integrate equation (3.11) in time (with the zero initial conditions), and then Fourier transform the result in space,  $\mathbf{r} \rightarrow \mathbf{k}$ . We have

$${}^1\tilde{B}_{\mathbf{k}\chi}(t) = ik_{\gamma}(\delta_{\chi\alpha}\delta_{\gamma\delta} - \delta_{\gamma\alpha}\delta_{\chi\delta}) \int_0^t \sum_{\mathbf{k}''=\mathbf{k}-\mathbf{k}'} {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^0\tilde{B}_{\mathbf{k}'\delta} dt'. \quad (4.26)$$

Here we also use the fact that the fluid velocity and the magnetic field are divergence free, and therefore,  $k'_{\alpha} {}^0\tilde{B}_{\mathbf{k}'\alpha} = 0$  and  $k''_{\alpha} {}^1\tilde{V}_{\mathbf{k}''\alpha} = 0$ . The complex conjugate of formula (4.26) is

$${}^1\tilde{B}_{\mathbf{k}\chi}^*(t) = -ik_{\tau}(\delta_{\chi\beta}\delta_{\tau\eta} - \delta_{\tau\beta}\delta_{\chi\eta}) \int_0^t \sum_{\mathbf{k}'''=\mathbf{k}-\mathbf{k}'''} {}^1\tilde{V}_{\mathbf{k}'''\beta}^*(t'') {}^0\tilde{B}_{\mathbf{k}'''\eta} dt''. \quad (4.27)$$

Using these two equations, we obtain

$$\begin{aligned} \langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle &= k_{\gamma}k_{\tau}(\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\tau\eta} - \delta_{\alpha\eta}\delta_{\beta\tau}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\tau\eta} + \delta_{\alpha\gamma}\delta_{\beta\tau}\delta_{\delta\eta}) \\ &\quad \times \int_0^t \int_0^t \sum_{\mathbf{k}''=\mathbf{k}-\mathbf{k}'} \sum_{\mathbf{k}'''=\mathbf{k}-\mathbf{k}'''} {}^0\tilde{B}_{\mathbf{k}'\delta} {}^0\tilde{B}_{\mathbf{k}'''\eta}^* \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}'''\beta}^*(t'') \rangle dt' dt'' \\ &= k^2 \hat{k}_{\gamma} \hat{k}_{\tau} (\delta_{\alpha\beta}\delta_{\gamma\delta}\delta_{\tau\eta} - \delta_{\alpha\eta}\delta_{\beta\tau}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\tau\eta} + \delta_{\alpha\gamma}\delta_{\beta\tau}\delta_{\delta\eta}) \\ &\quad \times \sum_{\mathbf{k}'=\mathbf{k}-\mathbf{k}'} \int_0^t \int_0^t {}^0b_{\delta\eta} |{}^0\tilde{B}_{\mathbf{k}'}|^2 \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle dt' dt''. \end{aligned} \quad (4.28)$$

Here, we use  $\langle {}^1\tilde{V}_{\mathbf{k}''\alpha} {}^1\tilde{V}_{\mathbf{k}''\beta}^* \rangle \propto \delta_{\mathbf{k}'', \mathbf{k}''}$ , see equation (3.81), and therefore,  $\mathbf{k}'' = \mathbf{k}'$  and  $\mathbf{k}''' = \mathbf{k}'$ . We also assume that  ${}^0b_{\alpha\beta} = \text{const}$  (our first working hypothesis), and there-

fore,  ${}^0\tilde{B}_{\mathbf{k}'\delta}{}^0\tilde{B}_{\mathbf{k}'\eta}^* = {}^0b_{\delta\eta}|{}^0\tilde{B}_{\mathbf{k}'}|^2$ . Now, according to definition (2.18) for the magnetic energy spectrum  $M(t, k)$  and equation (4.24), in order to obtain the ensemble averaged energy spectrum,  $\langle M(t, k) \rangle$ , we need to integrate  $\langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle$  over all directions of unit vector  $\hat{\mathbf{k}}$ . Equation (4.28) has four terms on the right-hand-side. Therefore, we have four terms for  $\langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle$  integrated over  $\hat{\mathbf{k}}$ ,

$$\int k^2 \langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^2\hat{\mathbf{k}} = \mathcal{T} - \mathcal{T}' - \mathcal{T}'' + \mathcal{T}''', \quad (4.29)$$

where

$$\mathcal{T} = k^4 \left( \frac{L}{2\pi} \right)^3 \int_0^\infty dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}'} \int \mu^2 d^2\hat{\mathbf{k}} \int_0^t \int_0^t \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'', \quad (4.30)$$

$$\mathcal{T}' = k^4 \left( \frac{L}{2\pi} \right)^3 \int_0^\infty dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}'} \int \mu d^2\hat{\mathbf{k}} \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle \hat{k}_\beta dt' dt'', \quad (4.31)$$

$$\mathcal{T}'' = \mathcal{T}'^* = \mathcal{T}', \quad (4.32)$$

$$\mathcal{T}''' = k^4 \left( \frac{L}{2\pi} \right)^3 \int_0^\infty dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}'} \int d^2\hat{\mathbf{k}} \int_0^t \int_0^t \hat{k}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle \hat{k}_\beta dt' dt'', \quad (4.33)$$

$$\mathbf{k}'' = \mathbf{k} - \mathbf{k}'. \quad (4.34)$$

Here,  $\mu = (\hat{\mathbf{k}} \cdot {}^0\hat{\mathbf{b}})$ , see equation (3.32), and we replace the summation over  $\mathbf{k}'$  by the integration over  $\mathbf{k}'$  (see Appendix A), which, in turn, is replaced by the double integration over  $k' = |\mathbf{k}'|$  and over  $\hat{\mathbf{k}}' = \mathbf{k}'/k'$ .

[In equations (4.29)–(4.33) and below, in order to shorten our notations, we use the same notation for a Fourier coefficient, no matter whether it is a discrete or a continuous function of  $\mathbf{k}$ . For example, the function  $\langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle$  in equation (4.28) is a discrete function of  $\mathbf{k}$ , while the function  $\langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle$  in equation (4.29), strictly speaking, is the appropriately defined continuous function of  $\mathbf{k}$ ,  $\langle |{}^1\tilde{\mathbf{B}}(t, \mathbf{k})|^2 \rangle$ , see Appendix A. However, it is convenient to use the same notation for the both functions.]

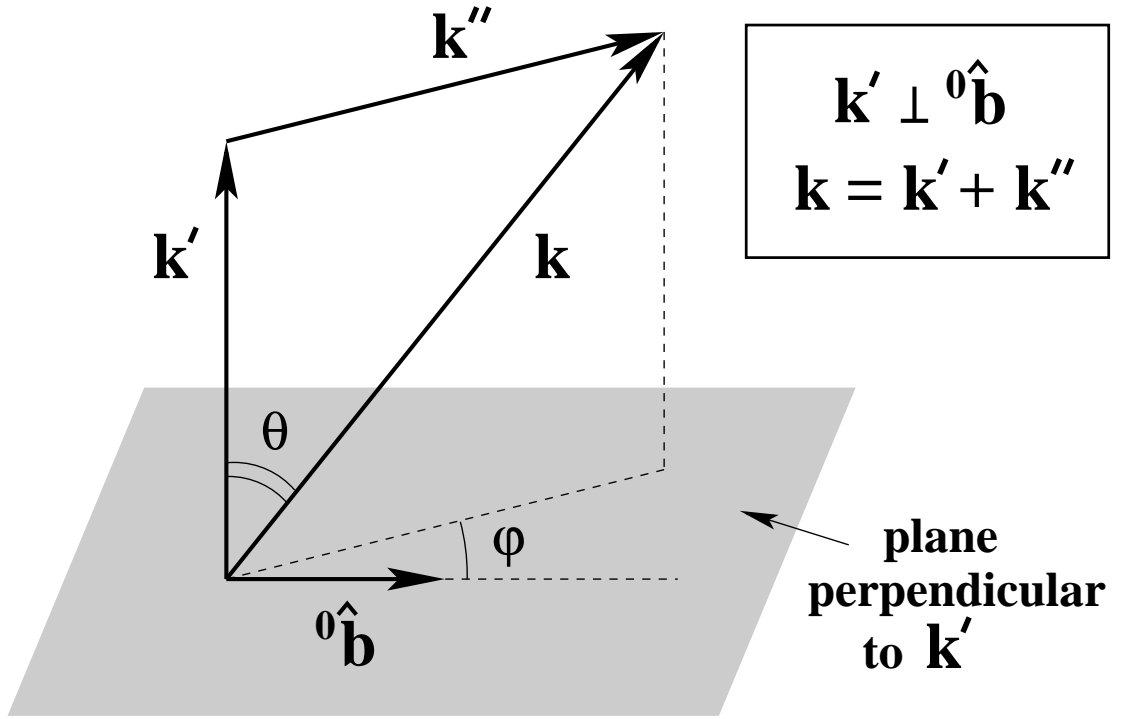


Figure 4.1: This plot shows relative position of vectors  $\hat{\mathbf{b}}$ ,  $\mathbf{k}$ ,  $\mathbf{k}'$  and  $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$  in space for the mode-coupling kernel (4.57). The  $k''$  modes of the turbulence interact with the  $k'$  modes of the magnetic field to change the energy in the  $k$  modes of the magnetic field. (In our case of an initially straight magnetic field vector  $\mathbf{k}'$  is perpendicular to  $\hat{\mathbf{b}}$  because the field is divergence free.)

Now, refer to Figure 4.1. Note that vector  $\mathbf{k}'$  is perpendicular to the zero order magnetic field unit vector  ${}^0\hat{\mathbf{b}}$  because the field is divergence free,  $k'_\alpha {}^0\tilde{B}_{\mathbf{k}'\alpha} = 0$ . We also have  $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$ , see equation (4.34). Therefore, the following useful equations are valid, see also Figure 4.1,

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' = \cos \theta, \quad (4.35)$$

$$k'' = |\hat{\mathbf{k}}''| = (k^2 + k'^2 - 2kk' \cos \theta)^{1/2}, \quad (4.36)$$

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'' = \frac{\hat{\mathbf{k}} \cdot \mathbf{k}''}{k''} = \frac{\hat{\mathbf{k}} \cdot (\mathbf{k} - \mathbf{k}')}{k''} = \frac{k - k' \cos \theta}{k''}, \quad (4.37)$$

$$\mu = \hat{\mathbf{k}} \cdot {}^0\hat{\mathbf{b}} = \sin \theta \cos \varphi, \quad (4.38)$$

$$\mu'' = {}^0\hat{\mathbf{b}} \cdot \hat{\mathbf{k}}'' = \frac{{}^0\hat{\mathbf{b}} \cdot \mathbf{k}''}{k''} = \frac{{}^0\hat{\mathbf{b}} \cdot \mathbf{k}}{k''} = \frac{k}{k''} \sin \theta \cos \varphi, \quad (4.39)$$

$$\begin{aligned} 1 - \mu''^2 &= \frac{k''^2 - k^2 \sin^2 \theta \cos^2 \varphi}{k''^2} = \frac{k^2 + k'^2 - 2kk' \cos \theta - k^2 \sin^2 \theta \cos^2 \varphi}{k''^2} \\ &= \frac{k'^2 - 2kk' \cos \theta + k^2 - k^2 \sin^2 \theta + k^2 \sin^2 \theta \sin^2 \varphi}{k''^2} \\ &= \frac{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi}{k''^2}, \end{aligned} \quad (4.40)$$

$$d^2\hat{\mathbf{k}} = \sin \theta \, d\theta \, d\varphi. \quad (4.41)$$

Using these formulas, equation (3.81), and equation (3.87) in the limit  $t \gg \tau$ , it is straightforward to calculate the double time integral terms in equations (4.30)–(4.33),

$$\int_0^t \int_0^t \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'' = J_{0k''} t \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \left[ 1 + \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \right], \quad (4.42)$$

$$\begin{aligned} \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle \hat{k}_\beta dt' dt'' &= J_{0k''} t \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \\ &\times \frac{k'(k' - k \cos \theta)}{k''^2} \sin \theta \cos \varphi \end{aligned} \quad (4.43)$$

$$\int_0^t \int_0^t \hat{k}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle \hat{k}_\beta dt' dt'' = J_{0k''} t \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2$$

$$\begin{aligned}
& \times \left[ \frac{k'^2 \sin^2 \theta \sin^2 \varphi}{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi} \right. \\
& + \left( 1 + \frac{2\Omega''}{\Omega''_{\text{rd}}} \right)^{-2} \frac{k'^2}{k''^2} \\
& \left. \times \frac{(k' - k \cos \theta)^2 \sin^2 \theta \cos^2 \varphi}{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi} \right]. \quad (4.44)
\end{aligned}$$

Here,  $\bar{\Omega}''$ ,  $\Omega''$  and  $\Omega''_{\text{rd}}$  depend on  $k''$  and on  $\mu''^2$ , see equations (3.37), (3.38) and (3.64). In turn,  $k''$  and on  $\mu''$  are functions of  $k$ ,  $k'$ ,  $\theta$  and  $\varphi$ , given by equations (4.36) and (4.39).

Now, we substitute formulas (4.38), (4.41) and (4.42)–(4.44) into equations (4.30), (4.31) and (4.33). The factors that we obtain in these equations after integration over  $d^2 \hat{\mathbf{k}} = \sin \theta \, d\theta \, d\varphi$  depend only on  $k$  and  $k'$ , so they can be exchanged with the integrations over  $d^2 \hat{\mathbf{k}}'$ . Combining the results together in equation (4.29), we obtain

$$\int k^2 \langle |^1 \tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^2 \hat{\mathbf{k}} = \int_0^\infty dk' \mathcal{K}_1(k, k') \int k'^2 |^0 \tilde{B}_{\mathbf{k}'}|^2 d^2 \hat{\mathbf{k}}', \quad (4.45)$$

where

$$\begin{aligned}
\mathcal{K}_1(k, k') = & t k^4 \left( \frac{L}{2\pi} \right)^3 \int_0^\pi d\theta \sin^3 \theta J_{0k''} \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}''}{\Omega''_{\text{rd}}} \right)^2 \\
& \times \left\{ \frac{k'^2 + 2k(k - k' \cos \theta) \cos^2 \varphi}{k''^2} - \left[ 1 - \left( 1 + \frac{2\Omega''}{\Omega''_{\text{rd}}} \right)^{-2} \right] \frac{k^2}{k''^2} \right. \\
& \left. \times \frac{(k' - k \cos \theta)^2 + (k^2 - k'^2) \sin^2 \theta \sin^2 \varphi}{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi} \cos^2 \varphi \right\}, \quad (4.46)
\end{aligned}$$

and  $k''$  is given by equation (4.36).

### 4.2.3 The $\langle {}^2\tilde{B}_{\mathbf{k}\alpha}(t) \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.}$ term

Next, we calculate the  $\langle {}^2\tilde{B}_{\mathbf{k}\alpha}(t) \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.}$  term of expansion equation (4.24). First, we integrate equation (3.18) in time (with the zero initial conditions), and then Fourier transform the result in space,  $\mathbf{r} \rightarrow \mathbf{k}$ . We have

$${}^2\tilde{B}_{\mathbf{k}\eta}(t) = ik_\tau(\delta_{\eta\beta}\delta_{\tau\chi} - \delta_{\tau\beta}\delta_{\eta\chi}) \int_0^t \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} [{}^1\tilde{V}_{\mathbf{k}''\beta}(t') {}^1\tilde{B}_{\mathbf{k}'\chi}(t') + {}^2\tilde{V}_{\mathbf{k}''\beta}(t') {}^0\tilde{B}_{\mathbf{k}'\chi}] dt', \quad (4.47)$$

where we use the divergence free conditions  $k_\alpha {}^0\tilde{B}_{\mathbf{k}\alpha} = 0$ ,  $k_\alpha {}^1\tilde{B}_{\mathbf{k}\alpha} = 0$ ,  $k_\alpha {}^1\tilde{V}_{\mathbf{k}\alpha} = 0$  and  $k_\alpha {}^2\tilde{V}_{\mathbf{k}\alpha} = 0$ . Second, we ensemble average his equation. The second term in the brackets [...] averages out because of equation (3.83). Then, we multiply the resulting equation by  ${}^0\tilde{B}_{\mathbf{k}\eta}^*$ , add the complex conjugate, and use formula (4.26) for  ${}^1\tilde{B}_{\mathbf{k}'\chi}$ . We have

$$\begin{aligned} \langle {}^2\tilde{B}_{\mathbf{k}\eta} \rangle {}^0\tilde{B}_{\mathbf{k}\eta}^* + \text{c.c.} &= ik_\tau(\delta_{\eta\beta}\delta_{\tau\chi} - \delta_{\tau\beta}\delta_{\eta\chi}) \int_0^t \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} \langle {}^1\tilde{V}_{\mathbf{k}''\beta}(t') {}^1\tilde{B}_{\mathbf{k}'\chi}(t') \rangle {}^0\tilde{B}_{\mathbf{k}\eta}^* dt' + \text{c.c.} \\ &= ik_\tau(\delta_{\eta\beta}\delta_{\tau\chi} - \delta_{\tau\beta}\delta_{\eta\chi}) i(\delta_{\chi\alpha}\delta_{\gamma\delta} - \delta_{\gamma\alpha}\delta_{\chi\delta}) \\ &\quad \times \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} \sum_{\substack{\mathbf{k}''' \\ \mathbf{k}^{\text{iv}} = \mathbf{k}' - \mathbf{k}'''}} k'_\gamma \int_0^t dt' \int_0^{t'} dt'' \langle {}^1\tilde{V}_{\mathbf{k}^{\text{iv}}\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle {}^0\tilde{B}_{\mathbf{k}\eta}^* {}^0\tilde{B}_{\mathbf{k}'''\delta} + \text{c.c.} \\ &= -k_\tau(\delta_{\alpha\tau}\delta_{\eta\beta}\delta_{\gamma\delta} - \delta_{\alpha\eta}\delta_{\tau\beta}\delta_{\gamma\delta} + \delta_{\delta\eta}\delta_{\gamma\alpha}\delta_{\tau\beta}) {}^0\tilde{B}_{\mathbf{k}\eta}^* {}^0\tilde{B}_{\mathbf{k}\delta} \\ &\quad \times \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} k'_\gamma \int_0^t dt' \int_0^{t'} dt'' \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle + \text{c.c.} \end{aligned} \quad (4.48)$$

Here, we use  $\langle {}^1\tilde{V}_{\mathbf{k}^{\text{iv}}\alpha} {}^1\tilde{V}_{\mathbf{k}''\beta} \rangle \propto \delta_{\mathbf{k}^{\text{iv}}, -\mathbf{k}''}$ , see equation (3.81), and therefore,  $\mathbf{k}^{\text{iv}} = -\mathbf{k}''$  and  $\mathbf{k}''' = \mathbf{k}$ . We also use the field divergence free condition,  $k_\alpha {}^0\tilde{B}_{\mathbf{k}\alpha} = 0$ , this is why we have only three terms at the end. Third, we change the summation over  $\mathbf{k}'$  to

summation over  $\mathbf{k}''$  in equation (4.48). We have

$$\begin{aligned}
\langle {}^2\tilde{B}_{\mathbf{k}\eta} \rangle {}^0\tilde{B}_{\mathbf{k}\eta}^* + \text{c.c.} &= -k_\tau (\delta_{\alpha\tau} \delta_{\eta\beta} \delta_{\gamma\delta} - \delta_{\alpha\eta} \delta_{\tau\beta} \delta_{\gamma\delta} + \delta_{\delta\eta} \delta_{\gamma\alpha} \delta_{\tau\beta}) {}^0\tilde{B}_{\mathbf{k}\eta}^* {}^0\tilde{B}_{\mathbf{k}\delta} \\
&\times \sum_{\mathbf{k}''} (k_\gamma - k_\gamma'') \int_0^t dt' \int_0^{t'} dt'' \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle + \text{c.c.} \\
&= -2k_\alpha k_\beta |{}^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 \sum_{\mathbf{k}''} \int_0^t dt' \int_0^{t'} dt'' \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \\
&\quad + 2k_\tau (\delta_{\alpha\tau} \delta_{\eta\beta} - \delta_{\alpha\eta} \delta_{\tau\beta}) {}^0b_{\eta\gamma} |{}^0\tilde{B}_{\mathbf{k}}|^2 \\
&\quad \times \sum_{\mathbf{k}''} k_\gamma'' \int_0^t dt' \int_0^{t'} dt'' \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \\
&= -2k^2 |{}^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 \sum_{\mathbf{k}''} \int_0^t dt' \int_0^{t'} dt'' \hat{k}_\alpha \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \hat{k}_\beta \\
&\quad + 2k |{}^0\tilde{B}_{\mathbf{k}}|^2 \sum_{\mathbf{k}''} {}^0\hat{b}_\gamma k_\gamma'' \int_0^t dt' \int_0^{t'} dt'' \hat{k}_\alpha \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle {}^0\hat{b}_\beta \\
&\quad - 2k |{}^0\tilde{B}_{\mathbf{k}}|^2 \sum_{\mathbf{k}''} {}^0\hat{b}_\gamma k_\gamma'' \int_0^t dt' \int_0^{t'} dt'' {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \hat{k}_\beta \\
&= -2k^2 |{}^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 \left( \frac{L}{2\pi} \right)^3 \int_{-\infty}^{\infty} d^3\mathbf{k}'' \\
&\quad \times \int_0^t dt' \int_0^{t'} dt'' \hat{k}_\alpha \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \hat{k}_\beta. \tag{4.49}
\end{aligned}$$

Here, we again used the divergence free conditions on the field and on the fluid velocities, and formula  ${}^0\tilde{B}_{\mathbf{k}\eta} {}^0\tilde{B}_{\mathbf{k}\delta}^* = {}^0b_{\eta\delta} |{}^0\tilde{B}_{\mathbf{k}}|^2$  ( ${}^0b_{\eta\delta} = \text{const}$ , according to our first working hypothesis). Two terms on the seventh and eighth lines of equation (4.49) cancel each other because of the symmetry of tensor  $\langle {}^1\tilde{V}_{-\mathbf{k}''\alpha} {}^1\tilde{V}_{\mathbf{k}''\beta} \rangle$  with respect to exchange  $\alpha \leftrightarrow \beta$  and  $\mathbf{k}'' \leftrightarrow -\mathbf{k}''$ , see equation (3.81). We change the summation over  $\mathbf{k}''$  to the integration over  $\mathbf{k}''$  in the last line of equation (4.49).

Now, we integrate  $\langle {}^2\tilde{B}_{\mathbf{k}\eta} \rangle {}^0\tilde{B}_{\mathbf{k}\eta}^* + \text{c.c.}$ , given by equation (4.49), over all directions

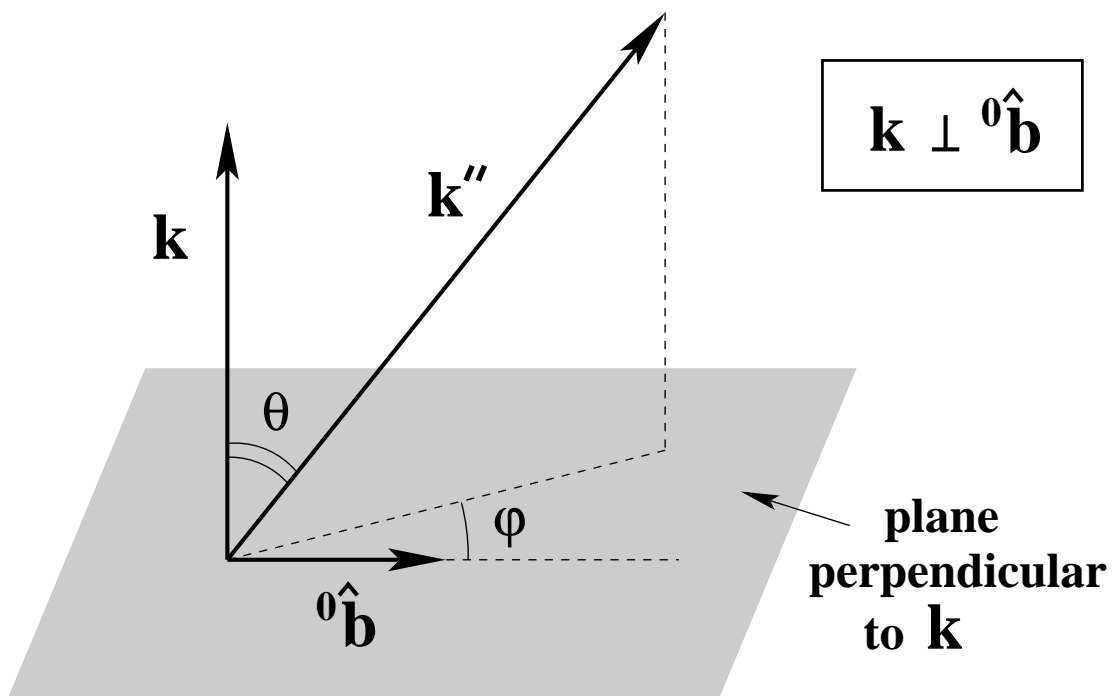


Figure 4.2: This plot shows relative position of vectors  ${}^0\hat{\mathbf{b}}$ ,  $\mathbf{k}$  and  $\mathbf{k}''$  in space for equations (4.50)–(4.55).

of unit vector  $\hat{\mathbf{k}}$ . We have

$$\begin{aligned} \int k^2 \left[ \langle {}^2\tilde{B}_{\mathbf{k}\eta} \rangle {}^0\tilde{B}_{\mathbf{k}\eta}^* + \text{c.c.} \right] d^2\hat{\mathbf{k}} = & -2k^2 \int k^2 |{}^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 d^2\hat{\mathbf{k}} \left( \frac{L}{2\pi} \right)^3 \int_{-\infty}^{\infty} d^3\mathbf{k}'' \\ & \times \int_0^t dt' \int_0^{t'} dt'' \hat{k}_\alpha \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \hat{k}_\beta. \end{aligned} \quad (4.50)$$

Let us refer to Figure 4.2. Note that vector  $\mathbf{k}$  is perpendicular to the zero order magnetic field unit vector  ${}^0\hat{\mathbf{b}}$  because the field is divergence free,  $k_\alpha {}^0\tilde{B}_{\mathbf{k}\alpha} = 0$ . We also have

$$\hat{\mathbf{k}}'' \cdot \hat{\mathbf{k}} = \cos \theta, \quad (4.51)$$

$$\mu'' = \hat{\mathbf{k}}'' \cdot {}^0\hat{\mathbf{b}} = \sin \theta \cos \varphi, \quad (4.52)$$

$$d^3\mathbf{k}'' = k''^2 dk'' \sin \theta d\theta d\varphi. \quad (4.53)$$

Using the first two of these equations, equation (3.81), and equation (3.86) in the limit  $t \gg \tau$ , we calculate the double time integral in equation (4.50),

$$\begin{aligned} \int_0^t dt' \int_0^{t'} dt'' \hat{k}_\alpha \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \hat{k}_\beta = & \frac{1}{2} J_{0k''} t \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \sin^2 \theta \\ & \times \left\{ 1 - \left[ 1 - \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \right] \right. \\ & \left. \times \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}. \end{aligned} \quad (4.54)$$

Here,  $\bar{\Omega}''$ ,  $\Omega''$  and  $\Omega_{\text{rd}}''$  depend on  $k''$  and on  $\mu''^2$ , see equations (4.52), (3.37), (3.38) and (3.64). Finally, substituting equation (4.54) into equation (4.50), and using

equation (4.53), we obtain

$$\begin{aligned}
\int k^2 \left[ \langle {}^2\tilde{B}_{\mathbf{k}\eta} \rangle {}^0\tilde{B}_{\mathbf{k}\eta}^* + \text{c.c.} \right] d^2\hat{\mathbf{k}} = & -t k^2 \int k^2 |{}^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 d^2\hat{\mathbf{k}} \\
& \times \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k''^2 J_{0k''} dk'' \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} d\varphi \\
& \times \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \left\{ 1 - \left[ 1 - \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \right] \right. \\
& \left. \times \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}. \tag{4.55}
\end{aligned}$$

#### 4.2.4 Collecting the terms together

Next, we substitute equations (4.24), (4.45) and (4.55) into equation (2.18) for the magnetic energy spectrum  $M(t, k)$ . We also make use of equation (4.25). We choose  $t$  small enough for the quasi-linear expansion to be valid, so that  $\partial_t \langle M(t, k) \rangle = [M(t, k) - M(0, k)]/t$ . As a result, we finally obtain *the mode coupling equation* for the magnetic energy spectrum,

$$\frac{\partial M}{\partial t} = \int_0^\infty K(k, k') M(t, k') dk' - 2 \frac{\eta_r}{4\pi} k^2 M(t, k). \tag{4.56}$$

Here the mode coupling kernel  $K(k, k')$  is

$$\begin{aligned}
K(k, k') = & k^4 \left( \frac{L}{2\pi} \right)^3 \int_0^\pi d\theta \sin^3 \theta J_{0k''} \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \\
& \times \left\{ \frac{k'^2 + 2k(k - k' \cos \theta) \cos^2 \varphi}{k''^2} - \left[ 1 - \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \right] \frac{k^2}{k''^2} \right. \\
& \left. \times \frac{(k' - k \cos \theta)^2 + (k^2 - k'^2) \sin^2 \theta \sin^2 \varphi}{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi} \cos^2 \varphi \right\}, \tag{4.57}
\end{aligned}$$

$$k'' = (k^2 + k'^2 - 2kk' \cos \theta)^{1/2}, \tag{4.58}$$

$$\frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} = 6 \frac{k''^2}{k_\nu^2}, \quad (4.59)$$

$$\frac{2\Omega''}{\Omega_{\text{rd}}''} = 90 \frac{k^2}{k_\nu^2} \frac{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi}{k''^2} \sin^2 \theta \cos^2 \varphi, \quad (4.60)$$

and the turbulent diffusion constant

$$\begin{aligned} \frac{\eta_T}{4\pi} &= \frac{1}{2} \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k''^2 J_{0k''} dk'' \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}'''}{\Omega_{\text{rd}}'''} \right)^2 \\ &\quad \times \left\{ 1 - \left[ 1 - \left( 1 + \frac{2\Omega'''}{\Omega_{\text{rd}}'''} \right)^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}, \end{aligned} \quad (4.61)$$

$$\frac{\bar{\Omega}'''}{\Omega_{\text{rd}}'''} = 6 \frac{k''^2}{k_\nu^2}, \quad (4.62)$$

$$\frac{2\Omega'''}{\Omega_{\text{rd}}'''} = 90 \frac{k''^2}{k_\nu^2} \sin^2 \theta \cos^2 \varphi (1 - \sin^2 \theta \cos^2 \varphi). \quad (4.63)$$

The function  $J_{0k}$  is given by equation (2.11). To derive the above formulas, we use equations (3.37), (3.38) and (3.64). To obtain equation (4.60) we use equations (4.39) and (4.40). To obtain equation (4.63), we use equation (4.52). In the last three equations for the turbulent diffusion constant we replace  $k''$  by  $k'''$  in order to distinguish it from  $k''$  in the equations for the coupling kernel.

If we consider the kinematic turbulent dynamo, then we need to take the limit  $\bar{\Omega} \rightarrow 0$ ,  $\Omega \rightarrow 0$ , or alternatively, the limit  $\Omega_{\text{rd}} \rightarrow \infty$  (see the footnote 4 on page 64). In this case, as one might expect, after integrating over  $\varphi$ , equation (4.57) reduces to equation (2.22), and after integrating over both  $\theta$  and  $\varphi$ , equation (4.61) reduces to equation (2.24).

## 4.3 The Magnetic Energy Spectrum on Subviscous Scales

Equations (4.56)–(4.63), which give the evolution of the magnetic energy spectrum in the magnetized turbulent dynamo theory, are the principal result of this thesis. However, these equations are rather complicated for easy interpretation. In this section we limit ourselves to the evolution of the magnetic energy spectrum on small subviscous scales. In this limit,  $k \gg k_\nu$ , and the integro-differential equation for the magnetic spectrum evolution simplifies to an ordinary differential equation. At the same time, the evolution of the magnetic energy spectrum on subviscous scales is of great interest for the origin of cosmic magnetic fields (see Chapter 5).

Let us refer to equation (4.57) for the mode coupling kernel  $K(k, k')$ . The function  $J_{0k''}$  cuts off at the viscous wave number  $k_\nu$  [see equation (2.11)]. Therefore, in the large- $k$  limit,  $k \gg k_\nu$ , we have  $k'' \sim |k - k'| \ll k, k'$ , and can expand the kernel  $K(k, k')$ . However, the simplest way of calculations is to introduce an arbitrary function of  $k$ ,  $F(k)$ , which varies slowly in the region  $k \gg k_\nu$ , and vanishes outside of this region [27]. To derive the mode coupling equation on small (subviscous) scales, we calculate the following integral

$$\begin{aligned}
\int_0^\infty F(k) \frac{\partial M}{\partial t} \Big|_{t=0} dk &= \frac{1}{4\pi\rho} \left( \frac{L}{2\pi} \right)^3 \int_{-\infty}^\infty F(k) \frac{\partial \langle |\tilde{\mathbf{B}}_{\mathbf{k}}|^2 \rangle}{\partial t} \Big|_{t=0} d^3\mathbf{k} \\
&= \frac{1}{4\pi\rho} \left( \frac{L}{2\pi} \right)^3 \int_{-\infty}^\infty F(k) \frac{\langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle - |\tilde{\mathbf{B}}_{\mathbf{k}}(0)|^2}{t} d^3\mathbf{k} \\
&= \frac{1}{4\pi\rho} \left( \frac{L}{2\pi} \right)^3 \frac{1}{t} \left\{ \int_{-\infty}^\infty F(k) \langle |^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^3\mathbf{k} \right. \\
&\quad \left. + \int_{-\infty}^\infty F(k) \left[ \langle |^2\tilde{B}_{\mathbf{k}\alpha}(t) \rangle ^0 \tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.} \right] d^3\mathbf{k} \right\}. \quad (4.64)
\end{aligned}$$

To obtain the first line of this equation, we use equation (2.18). To obtain the second line, we replace the time derivative by the time finite difference, assuming that  $t$  is small, and our quasi-linear expansion is valid. To obtain the final result in equation (4.64) [the third and the fourth lines], we use equations (4.24) and (4.25). Next, we use equations (4.29)–(4.34) and equation (4.50) to obtain the term in the brackets  $\{\dots\}$  in equation (4.64),

$$\begin{aligned}
& \int_{-\infty}^{\infty} F(k) \langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^3\mathbf{k} + \int_{-\infty}^{\infty} F(k) [\langle {}^2\tilde{B}_{\mathbf{k}\alpha}(t) \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.}] d^3\mathbf{k} \\
&= \int_0^{\infty} F(k) dk \int k^2 \langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^2\hat{\mathbf{k}} + \int_0^{\infty} F(k) dk \int k^2 [\langle {}^2\tilde{B}_{\mathbf{k}\alpha}(t) \rangle {}^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.}] d^2\hat{\mathbf{k}} \\
&= \left(\frac{L}{2\pi}\right)^3 [\mathcal{T}_{\diamond} - \mathcal{T}'_{\diamond} - \mathcal{T}''_{\diamond} + \mathcal{T}'''_{\diamond} + \mathcal{T}_{\diamond}^{\text{iv}}], \tag{4.65}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_{\diamond} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu^2 k^2 F(k) |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' d^3\mathbf{k} \int_0^t \int_0^t \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'' \\
&= \int_{-\infty}^{\infty} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} \mu''^2 k''^2 F(k) d^3\mathbf{k}'' \int_0^t \int_0^t \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'', \tag{4.66}
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}'_{\diamond} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu k F(k) |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' d^3\mathbf{k} \int_0^t \int_0^t {}^0\hat{b}_{\alpha} \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k_{\beta} dt' dt'' \\
&= \int_{-\infty}^{\infty} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} \mu'' k'' F(k) d^3\mathbf{k}'' \int_0^t \int_0^t {}^0\hat{b}_{\alpha} \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_{\beta} dt' dt'', \tag{4.67}
\end{aligned}$$

$$\mathcal{T}''_{\diamond} = \mathcal{T}'_{\diamond}, \tag{4.68}$$

$$\begin{aligned}
\mathcal{T}'''_{\diamond} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k) |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' d^3\mathbf{k} \int_0^t \int_0^t k_{\alpha} \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k_{\beta} dt' dt'' \\
&= \int_{-\infty}^{\infty} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} F(k) d^3\mathbf{k}'' \int_0^t \int_0^t k'_{\alpha} \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_{\beta} dt' dt'', \tag{4.69}
\end{aligned}$$

$$\mathbf{k} = \mathbf{k}' + \mathbf{k}'', \tag{4.70}$$

and

$$\begin{aligned}
\mathcal{T}_{\diamond}^{\text{iv}} &= -2 \int_{-\infty}^{\infty} F(k) |{}^0\tilde{B}_{\mathbf{k}}|^2 d^3\mathbf{k} \int_{-\infty}^{\infty} d^3\mathbf{k}'' \int_0^t dt' \int_0^{t'} dt'' k_{\alpha} \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle k_{\beta} \\
&= - \int_{-\infty}^{\infty} F(k) |{}^0\tilde{B}_{\mathbf{k}}|^2 d^3\mathbf{k} \int_{-\infty}^{\infty} d^3\mathbf{k}'' \int_0^t \int_0^t k_{\alpha} \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle k_{\beta} dt' dt''. \quad (4.71)
\end{aligned}$$

Here, to obtain the final results in equations (4.66)–(4.69), we change the integration over  $\mathbf{k}$  in these equations to integration over  $\mathbf{k}''$  by making use of formula (4.70). We use the divergence free condition on the fluid velocities,  $k_{\alpha}'' {}^1\tilde{V}_{\mathbf{k}''\alpha}$ , and therefore,  $k_{\alpha} {}^1\tilde{V}_{\mathbf{k}''\alpha} = k'_{\alpha} {}^1\tilde{V}_{\mathbf{k}''\alpha}$ . We also use the obvious formula

$$\mu k = {}^0\hat{\mathbf{b}} \cdot \mathbf{k} = {}^0\hat{\mathbf{b}} \cdot \mathbf{k}'' = k''({}^0\hat{\mathbf{b}} \cdot \hat{\mathbf{k}}'') = \mu'' k'' = k'' \sin \theta \cos \varphi, \quad (4.72)$$

see Figure 4.3. To obtain the final result in equation (4.71), we make use of equation (3.81) and of the first equality in equation (3.87).

We calculate  $\mathcal{T}_{\diamond}$ ,  $\mathcal{T}'_{\diamond}$ ,  $\mathcal{T}'''_{\diamond}$  and  $\mathcal{T}_{\diamond}^{\text{iv}}$  separately, up to the second order in  $k'' \ll k, k'$ .

First, following Kulsrud and Anderson [27], in equations (4.66)–(4.69) we expand the slowly varying function  $F(k)$  in  $k'' \ll k$  at point  $k'$  up to the second order. We have, see Figure 4.3,

$$\begin{aligned}
k &= [k'^2 + k''^2 + 2(\mathbf{k}' \cdot \mathbf{k}'')]^{1/2} = k' \left[ 1 + \frac{2(\mathbf{k}' \cdot \mathbf{k}'')}{k'^2} + \frac{k''^2}{k'^2} \right]^{1/2} \\
&= k' + \frac{(\mathbf{k}' \cdot \mathbf{k}'')}{k'} + \frac{1}{2} \left[ \frac{k''^2}{k'} - \frac{(\mathbf{k}' \cdot \mathbf{k}'')^2}{k'^3} \right] = k' + k'' \cos \theta + \frac{k''^2}{2k'} \sin^2 \theta, \quad (4.73)
\end{aligned}$$

$$\begin{aligned}
F(k) &= F(k') + \frac{dF}{dk'}(k - k') + \frac{1}{2} \frac{d^2F}{dk'^2}(k - k')^2 \\
&= F(k') + \frac{dF}{dk'} k'' \cos \theta + \frac{1}{2k'} \frac{dF}{dk'} k''^2 \sin^2 \theta + \frac{1}{2} \frac{d^2F}{dk'^2} k'^2 \cos^2 \theta, \quad (4.74)
\end{aligned}$$

Second, we calculate  $\mathcal{T}_{\diamond}$ , given by equation (4.66). Because  $\mu'' k''^2$  is of the second

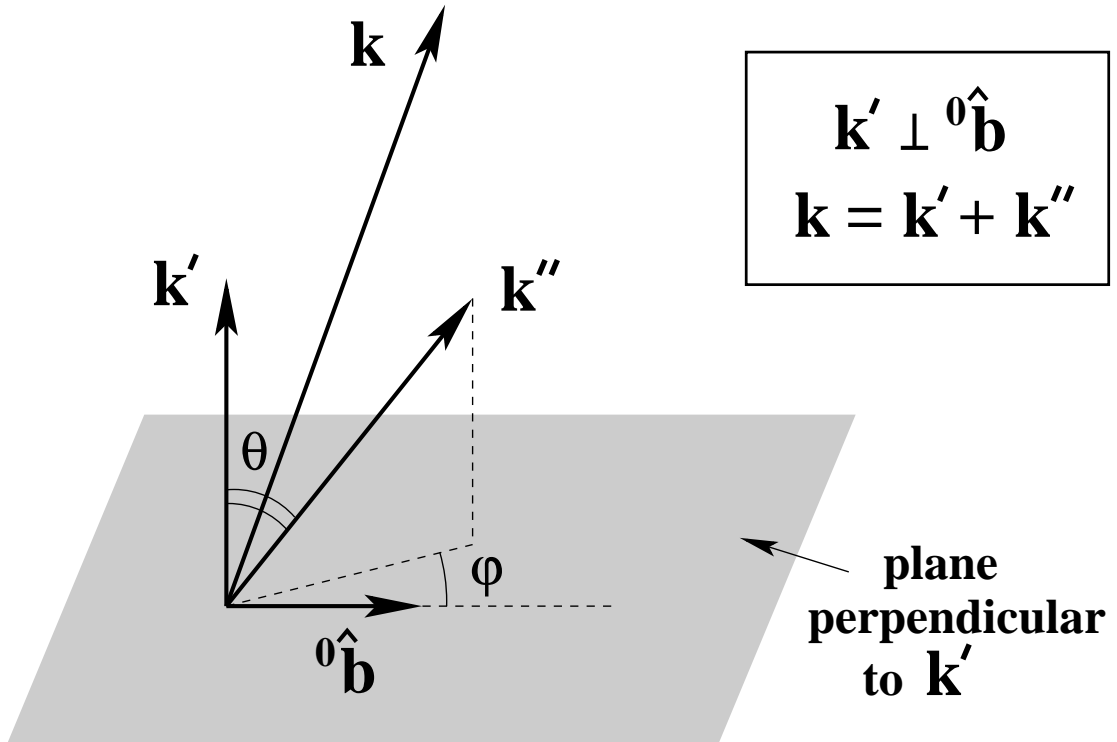


Figure 4.3: This plot shows relative position of vectors  $\hat{\mathbf{b}}$ ,  $\mathbf{k}'$ ,  $\mathbf{k}''$  and  $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$  in space for equations (4.66)–(4.70).

order in  $k''$ , we need to keep only the zero order term in expansion (4.74) for  $F(k)$ .

Thus, we have

$$\begin{aligned}
\mathcal{T}_\diamond &= \int_{-\infty}^{\infty} F(k') |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} \mu''^2 k''^2 d^3\mathbf{k}'' \int_0^t \int_0^t \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'' \\
&= t \int_0^{\infty} F(k') dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}}' \int_0^{\infty} k''^4 J_{0k''} dk'' \int_0^{\pi} d\theta \sin^3 \theta \\
&\quad \times \int_0^{2\pi} d\varphi \cos^2 \varphi \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \left[ 1 + \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \right], \tag{4.75}
\end{aligned}$$

where we use

$$d^3\mathbf{k}'' = k''^2 dk'' \sin \theta d\theta d\varphi, \tag{4.76}$$

and equations (4.42) and (4.72), see Figure 4.3. Here and below in this section, functions  $\bar{\Omega}''$ ,  $\Omega''$  and  $\Omega_{\text{rd}}''$  depend on  $k''$  and on  $\mu''^2 = \sin^2 \theta \cos^2 \varphi$ , see equations (3.37), (3.38) and (3.64).

Third, we calculate  $\mathcal{T}'_\diamond$ , given by equation (4.67). Because  $\mu''k''$  is of the first order in  $k''$ , we need to keep only the zero and the first order terms in expansion (4.74) for  $F(k)$ . We have

$$\begin{aligned}
\mathcal{T}'_\diamond &= \int_{-\infty}^{\infty} F(k') |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} \mu'' k'' d^3\mathbf{k}'' \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' \\
&\quad + \int_{-\infty}^{\infty} \frac{dF}{dk'} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} \mu'' k''^2 \cos \theta d^3\mathbf{k}'' \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt''. \tag{4.77}
\end{aligned}$$

Using equations (3.81), (3.87), (4.72) and Figure 4.3, we obtain

$$\int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' =$$

$$= -tk' J_{0k''} \sin \theta \cos \theta \cos \varphi \left(1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''}\right)^2 \left(1 + \frac{2\Omega''}{\Omega_{\text{rd}}''}\right)^{-2}. \quad (4.78)$$

Now, we substitute this equation and equation (4.76) into equation (4.77). The first term in equation (4.77) vanishes after the integration over  $\theta$  because the integrand is an odd function of  $\cos \theta$ .<sup>5</sup> The second term is nonzero. As a result, we have

$$\begin{aligned} \mathcal{T}'_{\diamond} = & -t \int_0^{\infty} k' \frac{dF}{dk'} dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\hat{\mathbf{k}}' \int_0^{\infty} k''^4 J_{0k''} dk'' \int_0^{\pi} d\theta \sin^3 \theta \cos^2 \theta \\ & \times \int_0^{2\pi} d\varphi \cos^2 \varphi \left(1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''}\right)^2 \left(1 + \frac{2\Omega''}{\Omega_{\text{rd}}''}\right)^{-2}. \end{aligned} \quad (4.79)$$

Fourth, we calculate  $\mathcal{T}'''_{\diamond}$ , given by equation (4.69). We need to keep all terms in expansion (4.74) for  $F(k)$ . We have

$$\begin{aligned} \mathcal{T}'''_{\diamond} = & \int_{-\infty}^{\infty} F(k') |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} d^3\mathbf{k}'' \int_0^t \int_0^t k'_{\alpha} \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_{\beta} dt' dt'' \\ & + \int_{-\infty}^{\infty} \frac{dF}{dk'} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} k'' \cos \theta d^3\mathbf{k}'' \int_0^t \int_0^t k'_{\alpha} \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_{\beta} dt' dt'' \\ & + \int_{-\infty}^{\infty} \frac{1}{2k'} \frac{dF}{dk'} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} k''^2 \sin^2 \theta d^3\mathbf{k}'' \int_0^t \int_0^t k'_{\alpha} \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_{\beta} dt' dt'' \\ & + \int_{-\infty}^{\infty} \frac{1}{2} \frac{d^2F}{dk'^2} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} k''^2 \cos^2 \theta d^3\mathbf{k}'' \int_0^t \int_0^t k'_{\alpha} \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_{\beta} dt' dt''. \end{aligned} \quad (4.80)$$

The first term in this equation is equal to minus  $\mathcal{T}_{\diamond}^{\text{iv}}$ , which is given by equation (4.71), [to see this, we simply change the notation in equation (4.71),  $\mathbf{k} \rightarrow \mathbf{k}'$ ]. To calculate the last three terms in equation (4.80), we again use equations (3.81), (3.87), (4.72)

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<sup>5</sup> This is because the Braginskii viscosity is invariant under reflection  $\hat{\mathbf{b}} \rightarrow -\hat{\mathbf{b}}$ . See also the footnote 10 on page 55.

and Figure 4.3 to obtain

$$\begin{aligned} & \int_0^t \int_0^t k'_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' \\ &= tk'^2 J_{0k''} \sin^2 \theta \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \left\{ 1 - \left[ 1 - \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}. \quad (4.81) \end{aligned}$$

[Compare this equation with equation (4.54) and Figure 4.3 with Figure 4.2]. Now, we substitute this equation and equation (4.76) into the last three terms of equation (4.80). The second term of equation (4.80) vanishes after the integration over  $\theta$  because the integrand is an odd function of  $\cos \theta$ . The last two terms of equation (4.80) are nonzero. As a result, we have

$$\begin{aligned} \mathcal{T}_\diamond''' &= -\mathcal{T}_\diamond^{\text{iv}} \\ &+ \frac{t}{2} \int_0^\infty k' \frac{dF}{dk'} dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2 \hat{\mathbf{k}}' \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^5 \theta \\ &\times \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \left\{ 1 - \left[ 1 - \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\} \\ &+ \frac{t}{2} \int_0^\infty k'^2 \frac{d^2 F}{dk'^2} dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2 \hat{\mathbf{k}}' \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \cos^2 \theta \\ &\times \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''} \right)^2 \left\{ 1 - \left[ 1 - \left( 1 + \frac{2\Omega''}{\Omega_{\text{rd}}''} \right)^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}. \quad (4.82) \end{aligned}$$

Now, we substitute equations (4.68), (4.75), (4.79) and (4.82) into equation (4.65). The first term of equation (4.82) cancels the  $\mathcal{T}_\diamond^{\text{iv}}$  term in equation (4.65). Then, we substitute the result that we get in equation (4.65) into formula (4.64), and using equation (2.18) for  $M(0, k')$ , we obtain

$$\int_0^\infty F(k) \left. \frac{\partial M}{\partial t} \right|_{t=0} dk = \left( \frac{L}{2\pi} \right)^3 \int_0^\infty \left[ \lambda_0 F(k') + \lambda_1 k' \frac{dF}{dk'} + \lambda_2 k'^2 \frac{d^2 F}{dk'^2} \right] M(0, k') dk', \quad (4.83)$$

where

$$\lambda_0 = \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi \cos^2 \varphi \left(1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''}\right)^2 \left[1 + \left(1 + \frac{2\Omega''}{\Omega_{\text{rd}}''}\right)^{-2}\right], \quad (4.84)$$

$$\begin{aligned} \lambda_1 = \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi \left(1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''}\right)^2 & \left\{ 2 \cos^2 \theta \cos^2 \varphi + \frac{1}{2} \sin^2 \theta \right. \\ & \left. - \left[1 - \left(1 + \frac{2\Omega''}{\Omega_{\text{rd}}''}\right)^{-2}\right] \left(2 + \frac{1}{2} \frac{\sin^2 \theta}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi}\right) \cos^2 \theta \cos^2 \varphi \right\}, \quad (4.85) \end{aligned}$$

$$\begin{aligned} \lambda_2 = \frac{1}{2} \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \cos^2 \theta & \times \int_0^{2\pi} d\varphi \left(1 + \frac{\bar{\Omega}''}{\Omega_{\text{rd}}''}\right)^2 \left\{ 1 - \left[1 - \left(1 + \frac{2\Omega''}{\Omega_{\text{rd}}''}\right)^{-2}\right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}. \quad (4.86) \end{aligned}$$

Next, we integrate the right-hand-side of equation (4.83) by parts over some extent in  $k'$  and choose  $F(k')$ , so that it and its derivative  $dF/dk'$  vanish at the end points.

We have

$$\int_0^\infty F \frac{\partial M}{\partial t} dk = \left(\frac{L}{2\pi}\right)^3 \int_0^\infty \left[ (\lambda_0 - \lambda_1 + 2\lambda_2)M + (4\lambda_2 - \lambda_1)k \frac{\partial M}{\partial k} + \lambda_2 k^2 \frac{\partial^2 M}{\partial k^2} \right] F dk. \quad (4.87)$$

This equation is valid for an arbitrary function  $F$ . As a result, the integrands on the left- and right-hand-side of this equation should be equal, and we finally obtain the mode-coupling equation for the magnetic energy spectrum  $M(t, k)$  on small (subviscous) scales

$$\frac{\partial M}{\partial t} = \frac{\Gamma}{5} \left[ k^2 \frac{\partial^2 M}{\partial k^2} - (\Lambda_1 - 1)k \frac{\partial M}{\partial k} + \Lambda_0 M \right], \quad (4.88)$$

where

$$\begin{aligned} \Gamma = & \frac{5}{2} \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} dk \int_0^\pi d\theta \sin^3 \theta \cos^2 \theta \\ & \times \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}} \right)^2 \left\{ 1 - \left[ 1 - \left( 1 + \frac{2\Omega}{\Omega_{\text{rd}}} \right)^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}, \end{aligned} \quad (4.89)$$

$$\begin{aligned} \Lambda_1 = & -3 + \frac{5}{\Gamma} \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} dk \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}} \right)^2 \\ & \times \left\{ 2 \cos^2 \theta \cos^2 \varphi + \frac{1}{2} \sin^2 \theta - \left[ 1 - \left( 1 + \frac{2\Omega}{\Omega_{\text{rd}}} \right)^{-2} \right] \right. \\ & \times \left. \left( 2 + \frac{1}{2} \frac{\sin^2 \theta}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right) \cos^2 \theta \cos^2 \varphi \right\}, \end{aligned} \quad (4.90)$$

$$\begin{aligned} \Lambda_0 = & 2 + \frac{5}{\Gamma} \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} dk \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}} \right)^2 \\ & \times \left\{ 2 \sin^2 \theta \cos^2 \varphi - \frac{1}{2} \sin^2 \theta + \left[ 1 - \left( 1 + \frac{2\Omega}{\Omega_{\text{rd}}} \right)^{-2} \right] \right. \\ & \times \left. \left( \cos^2 \theta - \sin^2 \theta + \frac{1}{2} \frac{\sin^2 \theta \cos^2 \theta}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right) \cos^2 \varphi \right\}, \end{aligned} \quad (4.91)$$

$$\frac{\bar{\Omega}}{\Omega_{\text{rd}}} = 6 \frac{k^2}{k_\nu^2}, \quad (4.92)$$

$$\frac{2\Omega}{\Omega_{\text{rd}}} = 90 \frac{k^2}{k_\nu^2} \sin^2 \theta \cos^2 \varphi (1 - \sin^2 \theta \cos^2 \varphi), \quad (4.93)$$

and function  $J_{0k}$  is given by equation (2.11). Here, we drop double primes, "", and use equations (4.84)–(4.86) and equations (3.37), (3.38), (3.64). We calculate  $\Gamma$  and dimensionless number  $\Lambda_0$  and  $\Lambda_1$  numerically in Appendix (D), and obtain

$$\Gamma \approx 100 \left( \frac{U_0 L}{\nu} \right)^{1/2} \frac{U_0}{L} \quad (4.94)$$

$$\Lambda_1 \approx 2, \quad (4.95)$$

$$\Lambda_0 \approx 5. \quad (4.96)$$

Therefore,

$$\frac{\partial M}{\partial t} = 20 \left( \frac{U_0 L}{\nu} \right)^{1/2} \frac{U_0}{L} \left[ k^2 \frac{\partial^2 M}{\partial k^2} - k \frac{\partial M}{\partial k} + 5M \right]. \quad (4.97)$$

It is interesting to compare it with equation (2.25), obtained by Kulsrud and Anderson in the kinematic dynamo case [27].

Now, assume that  $M(t, k_{\text{ref}})$  is known as a function of time at some reference wave number  $k = k_{\text{ref}}$ , then the solution of (4.88) is

$$M(t, k) = \int_{-\infty}^t M(t', k_{\text{ref}}) G(k/k_{\text{ref}}, t - t') dt', \quad (4.98)$$

where the Green's function  $G(k, t)$  is

$$\begin{aligned} G(k, t) &= \left( \frac{5}{4\pi} \right)^{1/2} \frac{k^{\Lambda_1/2} \ln k}{\Gamma^{1/2} t^{3/2}} e^{(\Gamma/5)(\Lambda_0 - \Lambda_1^2/4)t} e^{-5 \ln^2 k / 4\Gamma t} \\ &= \left( \frac{5}{4\pi} \right)^{1/2} \frac{k \ln k}{\Gamma^{1/2} t^{3/2}} e^{(4\Gamma/5)t} e^{-5 \ln^2 k / 4\Gamma t}. \end{aligned} \quad (4.99)$$

We derive these two equations in Appendix E, and use equations (4.95) and (4.96) for  $\Lambda_1$  and  $\Lambda_0$ . We see that a “signal”  $M(t, k_{\text{ref}})$ , at zero time, will increase exponentially as  $e^{(4/5)\Gamma t}$  and will extend down to the scale  $k_{\text{peak}} \approx e^{(4/5)\Gamma t} k_{\text{ref}}$ , where  $k_{\text{peak}}$  is the peak of function  $kG(k, t)$ , (of course, the field scale can not become less than the resistivity scale). As a result, in the magnetized dynamo theory the magnetic energy tends to quickly propagate to very small subviscous scales, the same way as it does in the kinematic dynamo theory, (this propagation is checked by the resistivity). However, the tail of the magnetic energy spectrum on  $k_{\text{ref}} \lesssim k \lesssim k_{\text{peak}}$  scales increases with the wavenumber as  $\propto k$  instead of  $\propto k^{3/2}$  in the kinematic theory. Note, that according to equations (4.20) and (4.94), the growth rate of the Green's function,  $(4/5)\Gamma$ , is

approximately equal to a half of the growth rate of the total magnetic energy,  $2\gamma$ .

In the end of this section, let us consider the kinematic turbulent dynamo. In this case we need to take the limit  $\bar{\Omega} \rightarrow 0$ ,  $\Omega \rightarrow 0$ , or alternatively, the limit  $\Omega_{\text{rd}} \rightarrow \infty$  in equations (4.88)–(4.91) [see the footnote 4 on page 64]. After integrating over  $\varphi$  and  $\theta$ , we have  $\Gamma = \gamma_o$ , where  $\gamma_o$  is the Kulsrud-Anderson magnetic energy growth rate, given by equation (2.17),  $\Lambda_1 = 3$  and  $\Lambda_0 = 6$ . As a result, as one might expect, in the kinematic dynamo case equation (4.88) reduces to equation (2.25), and the Green's function (4.99) reduces to formula (2.27).

## Chapter 5

# Discussion and Conclusions

As we discussed in the introductory section, the origin of galactic and extragalactic magnetic fields is one of the key questions in astrophysics. This thesis is devoted to this question. There are two prevailing theories for the origin of cosmic magnetic fields. First, the galactic turbulent dynamo theory, which states that the magnetic fields have been primarily amplified in differentially rotating galactic disks after the galaxies had been formed. Second, the primordial turbulent dynamo theory, which states that the fields have primarily been produced in protogalaxies, undergoing gravitational collapse. It seems that both observational and theoretical results favor the second, the primordial dynamo theory.

In calculations of the magnetic field evolution the previous dynamo theories assumed the regular *isotropic viscosity* for the turbulent plasma motions, and this is justified because of neutrals. However, in protogalaxies the temperature is so high, that there are no neutrals, and the viscosity is dominated by ions. Therefore, in a protogalaxy, as the magnetic field strength grows in time because of the dynamo inductive action, starting from its initial seed value, the plasma quickly becomes strongly magnetized, the viscosity becomes *the Braginskii tensor viscosity*, and the

turbulent motions on the viscous scales become strongly altered from the isotropic case. As a result, the turbulent dynamo becomes *the magnetized turbulent dynamo*.

In this thesis we have developed a theoretical basis for the magnetized turbulent dynamo, which operates in protogalaxies. The results of the kinematic dynamo theory already seem to support the primordial (protogalactic) dynamo origin of cosmic magnetic fields [28]. The results that we have obtained for the magnetized dynamo, further support this primordial origin theory. This is because the number of e-foldings of the total magnetic energy during the collapse of a protogalaxy, given by equation (4.22), is as much as ten times larger than that in the kinematic dynamo theory. Therefore, the number of e-foldings in the magnetized dynamo is more than large enough for the magnetic fields in protogalaxies to grow from their seed value, provided by the Biermann battery, up to the field-turbulence energy equipartition value. The number of e-foldings of the magnetic energy on the viscous scale, which is equal to the growth rate  $(4/5)\Gamma$  of the Green's function (4.99), is less by one half, but it is still sufficiently large <sup>1</sup>.

Another of our predictions is that the tail of the magnetic energy spectrum on the small subviscous scales increases with the wavenumber as  $\propto k$  [see the Green's function (4.99)], instead of  $\propto k^{3/2}$  in the kinematic theory. Therefore, in the magnetized dynamo the magnetic energy is slightly shifted to larger scales as compared to the kinematic dynamo case.

Although the important results of this thesis are convincing, we left out several issues in our theory. Therefore, let us itemize and briefly discuss the possibilities of

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<sup>1</sup> Of course, our results (4.17)–(4.23) for the magnetic energy growth rate are sensitive to the value of the physical parameter  $\Omega_{\text{rd}}$ , which is estimated in equation (3.64). We also left out the finite time correlation effects (see discussion below). Therefore, our result for the number of magnetic energy e-foldings should be viewed as an estimate, valid within a factor of order two. However, it is important that the number of e-foldings that we found in the magnetized dynamo theory is large and is clearly larger than that in the kinematic dynamo theory.

further research on the magnetized turbulent dynamos.

- **Calculations of magnetic field curvature.**

In our calculations in this thesis we assumed that as far as we are interested in the evolution of magnetic energy, we can consider the magnetic field lines to be initially straight. This our first working hypothesis. It relies on the assumption that in the case of the magnetized turbulent dynamo the magnetic field has a folding structure similar to the one that exists in the case of the kinematic turbulent dynamo [31, 46]. Unfortunately, the calculation methods employed in this thesis are not adequate to justify this our hypothesis because of complications that arise when one calculates the statistics of the field curvature. It would be interesting either to expand our theory to the case of curved magnetic field lines, or to carry out numerical simulations in order to check the field folding structure in the magnetized dynamo theory.

- **Effective rotational damping of turbulent velocities.**

Another hypothesis that we made, but did not proved, in this thesis was the inclusion of the effective rotational damping into our equations as a linear damping term. Basically, the rotational damping rate  $\Omega_{\text{rd}}$ , which is in front of this linear damping term [see equation (3.65)], enters our final results as a physical parameter. We obtained only an estimate of this parameter, so our final numerical results depend on its value. As we said above, the rotational damping is associated with non-linear coupling of velocity modes and is very important. It restricts the unlimited growth of velocity modes which are perpendicular to the magnetic field lines and are undamped by the Braginskii viscous forces. It is important to further theoretically study the mechanisms which stop the unlimited growth of the undamped velocities, and to carry out MHD numerical

simulations, including the Braginskii viscosity.

- **Finite time correlation effects.**

In this thesis we used the quasi-linear (up to the second order) expansion procedure in time to solve the MHD equations. We assumed that we can choose our time expansion parameter  $t$  small enough for the quasi-linear expansion to be valid, but large enough for it to be much larger than the turbulent eddy decorrelation time. This is equivalent to the assumption that the turbulent velocities are  $\delta$ -correlated in time. It is known from the kinematic turbulent dynamo theory that the effects of finite velocity correlation time can decrease the magnetic energy growth rate by a factor of order two [47]. Thus, including the finite time correlation effects into the magnetized dynamo theory is important but not vital, since our calculation predicts a very large number of e-foldings of the magnetic field strength.

- **Energy equipartition and the inverse cascade.**

Finally, the Green's function solution of the mode-coupling equation for the magnetic energy spectrum on subviscous scales indicates that in the magnetized dynamo theory the magnetic energy tends to quickly propagate to very small subviscous scales, similar to the kinematic dynamo case. On the other hand, the observed cosmic fields have rather large correlation lengths. Therefore, the magnetic field lines must be unwrapped on small scales by the Lorentz tension forces, while the field energy is transferred and amplified on larger scales during the inverse cascade stage. This stage happens when the magnetic field energy is comparable to the turbulent kinetic energy, i. e. when there is the energy equipartition between the field and the turbulence. Thus, calculations and/or numerical simulations of the inverse cascade with the Braginskii viscosity are

extremely important for the theory of the origin of cosmic magnetic fields. In the end of this chapter let us discuss the importance of the Braginskii viscosity for the inverse cascade in more details.

As we said in the first introductory chapter of this thesis, the theory of the inverse cascade in a plasma with the regular isotropic viscosity has a difficulty of unwrapping of the small-scale magnetic field lines. This difficulty can be understood as follows [12]. The equation on the turbulent velocities  $\mathbf{V}$  in the plasma with the isotropic viscosity, including the Lorentz forces, is

$$\partial_t V_\alpha = -P_{,\alpha} + f_\alpha + \nu \Delta V_\alpha + \frac{1}{4\pi\rho} (\mathbf{B} \cdot \nabla) B_\alpha - (\mathbf{V} \cdot \nabla) V_\alpha. \quad (5.1)$$

[Compare this equation to equation (2.31)]. The  $(1/4\pi\rho)(\mathbf{B} \cdot \nabla) B_\alpha$  term is the magnetic tension force, normalized to the plasma density  $\rho$ . The magnetic pressure term is included into the hydrodynamic pressure term,  $-P_{,\alpha}$ . The  $\nu \Delta V_\alpha$  term is the viscous term. We can estimate the velocity of the magnetic field lines unwrapping,  $V_{\text{unwrap}}$ , by Fourier transforming equation (5.1) in space,  $\mathbf{r} \rightarrow \mathbf{k}$ , and then balancing the viscous and the inertial forces against the magnetic tension force. For the isotropic case, the viscosity dominates on small scales, and we have

$$\nu k^2 V_{\text{unwrap}} \sim \frac{1}{4\pi\rho} k_\parallel B^2, \quad (5.2)$$

$$V_{\text{unwrap}} \sim \frac{k_\parallel}{k} \frac{k_\nu}{k} \frac{V_A^2}{\nu k_\nu} \ll V_A, \quad (5.3)$$

where  $V_A$  is the Alfven speed. The Alfven speed  $V_A \sim \nu k_\nu$  at the time of the field-turbulence energy equipartition on the viscous scale, and  $V_A < \nu k_\nu$  before the equipartition. If the field lines have a folding pattern (and they do in the kinematic dynamo theory), then  $k \gg k_\parallel \sim k_\nu$ , see Figure 2.1, and therefore, the unwrapping veloc-

ity (5.3) is small compared to the Alfvén speed, even at the equipartition. In other words, since  $V_A \sim V$  at the energy equipartition, then  $k_{\parallel} V_A \sim \gamma$  ( $\gamma$  is the magnetic energy growth rate), and the unwrapping rate  $k_{\parallel} V_{\text{unwrap}}$  is much smaller than  $\gamma$ . This means that the field continues to grow on the viscous and subviscous scales even at the equipartition.

However, in the case of the magnetized turbulent dynamo, the viscosity term in the above equation is modified and the anti-unwrapping argument does not apply. Indeed, the field unwrapping velocity is parallel and varies perpendicularly to the magnetic field lines. The large velocity gradient perpendicular to the magnetic field lines, which leads to a large perpendicular stress in the isotropic viscosity case, is irrelevant in the case of the Braginskii viscous forces (because the transfer of the ion momentum in the perpendicular direction is inhibited). Therefore, in the magnetized dynamo theory  $k_{\parallel} V_{\text{unwrap}} \sim k_{\parallel} V_A \sim \gamma$  at the equipartition, and the magnetic field strength saturates on the viscous and subviscous scales. As a result, the Braginskii viscosity makes the inverse cascade of the magnetic energy more likely, because the larger turbulent eddies can not deliver their energy to the magnetic field on the viscous and subviscous scales [26].

# Appendix A

## Fourier and Laplace Transformations

Let us consider an arbitrary continuous function  $f(t, \mathbf{r})$ , where  $t$  is time and  $\mathbf{r}$  is position in space. We assume that  $f$  is defined inside a spatial box of length  $L$  and, in the rest of space, is periodic with the box size. Then  $f(t, \mathbf{r})$  can be expressed in terms of its Fourier components  $\tilde{f}(t, \mathbf{k})$  as

$$f(t, \mathbf{r}) = \sum_{\mathbf{k}} \tilde{f}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}}. \quad (\text{A.1})$$

Here the summation is done over discrete values of vector  $\mathbf{k}$ , so that the  $x$ -,  $y$ - and  $z$ - components of  $\mathbf{k}$  are discrete and are given by equations

$$k_x(n_x) = \frac{2\pi}{L} n_x, \quad (\text{A.2})$$

$$k_y(n_y) = \frac{2\pi}{L} n_y, \quad (\text{A.3})$$

$$k_z(n_z) = \frac{2\pi}{L} n_z, \quad (\text{A.4})$$

where  $n_x$ ,  $n_y$  and  $n_z$  are arbitrary integer numbers. The Fourier coefficients  $\tilde{f}_{\mathbf{k}}(t)$  are given by the standard discrete Fourier transformation formula

$$\tilde{f}_{\mathbf{k}}(t) = \frac{1}{L^3} \int_{-L/2}^{L/2} f(t, \mathbf{r}) e^{-i\mathbf{k}\mathbf{r}} d^3\mathbf{r}. \quad (\text{A.5})$$

Now let us introduce a function  $\tilde{f}(t, \mathbf{k}')$ , which is a continuous function of the wave number  $\mathbf{k}'$ ,

$$\begin{aligned} \tilde{f}(t, \mathbf{k}') \stackrel{\text{def}}{=} \tilde{f}_{\mathbf{k}}(t) \quad \text{if} \quad & k_x(n_x) - \pi L^{-1} \leq k'_x < k_x(n_x) + \pi L^{-1}, \\ & k_y(n_y) - \pi L^{-1} \leq k'_y < k_y(n_y) + \pi L^{-1}, \\ & k_z(n_z) - \pi L^{-1} \leq k'_z < k_z(n_z) + \pi L^{-1}. \end{aligned} \quad (\text{A.6})$$

This continuous function is constant over small three-dimensional cubic volume elements  $(2\pi/L)^3$  in  $\mathbf{k}$ -space, and it is equal to the values of the discrete function  $\tilde{f}_{\mathbf{k}}(t)$  at the center points of these volume elements<sup>1</sup>. As a result, the Fourier series (A.1) can be rewritten, using the integration of this appropriately defined continuous function  $\tilde{f}(t, \mathbf{k})$  over  $\mathbf{k}$ , as

$$f(t, \mathbf{r}) = \sum_{\mathbf{k}} \tilde{f}_{\mathbf{k}}(t) e^{i\mathbf{k}\mathbf{r}} = \left(\frac{L}{2\pi}\right)^3 \int_{-\infty}^{\infty} \tilde{f}(t, \mathbf{k}) e^{i\mathbf{k}\mathbf{r}} d^3\mathbf{k}. \quad (\text{A.7})$$

In this thesis we frequently use the appropriately defined functions that are continuous in  $\mathbf{k}$ , instead of discrete Fourier components.

We can further Fourier transform functions  $\tilde{f}_{\mathbf{k}}(t)$  and  $\tilde{f}(t, \mathbf{k})$  in time, by making use of the standard continuous Fourier transformation  $t \rightarrow \omega$ . For example, for

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<sup>1</sup> Note, that the important property  $\tilde{f}(t, -\mathbf{k}) = \tilde{f}^*(t, \mathbf{k})$  is an exact consequence of  $\tilde{f}_{-\mathbf{k}} = \tilde{f}_{\mathbf{k}}^*$  everywhere in  $\mathbf{k}$ -space but on the faces of the cubic volume elements. This is not a problem though, because the faces occupy an infinitely small three-dimensional volume.

function  $\tilde{f}_{\mathbf{k}}(t)$  we have

$$\tilde{f}_{\mathbf{k}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}_{\mathbf{k}}(t) e^{i\omega t} dt, \quad (\text{A.8})$$

$$\tilde{f}_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}_{\mathbf{k}}(\omega) e^{-i\omega t} d\omega. \quad (\text{A.9})$$

We can also form the continuous Laplace transformation in time,  $t \rightarrow s$ ,

$$\tilde{f}_{\mathbf{k}}(s) = \int_0^{\infty} \tilde{f}_{\mathbf{k}}(t) e^{-st} dt, \quad (\text{A.10})$$

$$\widetilde{\frac{\partial \tilde{f}_{\mathbf{k}}}{\partial t}}(s) = s \tilde{f}_{\mathbf{k}}(s) - \tilde{f}_{\mathbf{k}}(t=0), \quad (\text{A.11})$$

$$\tilde{f}_{\mathbf{k}}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}_{\mathbf{k}}(s) e^{ts} ds. \quad (\text{A.12})$$

In the last formula all poles of function  $\tilde{f}_{\mathbf{k}}(s)$  should be to the left of the integration contour chosen in the complex plane.

## Appendix B

### The Ensemble Average of the Second Order Velocities, $\langle {}^2\tilde{V}_{\mathbf{k}\alpha}(t) \rangle$

To derive equation (3.52) for the ensemble averaged second order velocity in our case of an initially straight field  ${}^0b_{\alpha\beta} = \text{const}$ , we proceed as follows. Let choose a system of coordinates in which the initial field  ${}^0\hat{\mathbf{b}}$  is along the  $x$ -direction. In this case  ${}^0b_{\alpha\beta} = \delta_{\alpha x}\delta_{\beta x}$ , and we rewrite equation (3.20) as

$$\begin{aligned} \partial_t {}^2v_\alpha &= -{}^2P_{,\alpha} + 3\nu\delta_{\alpha x}{}^2v_{x,xx} + 3\nu[{}^1b_{\alpha\beta}({}^1v_{x,x} + U_{x,x})]_{,\beta} \\ &+ 3\nu\delta_{\alpha x}[{}^1b_{\mu\nu}({}^1v_{\mu,\nu} + U_{\mu,\nu})]_{,x} - [{}^1v_\alpha U_\beta - U_\alpha {}^1v_\beta - {}^1v_\alpha {}^1v_\beta]_{,\beta}, \quad (\text{B.1}) \end{aligned}$$

Here, we use formula  ${}^1b_{\alpha\beta\mu\nu} = {}^1b_{\alpha\beta}{}^0b_{\mu\nu} + {}^0b_{\alpha\beta}{}^1b_{\mu\nu}$ , see definitions (2.36)–(2.38). Now, first, we Fourier transform this equation in space,  $\mathbf{r} \rightarrow \mathbf{k}$ . Second, we multiply the transformed equation on the left by the tensor  $\delta_{\gamma\alpha}^\perp = \delta_{\gamma\alpha} - \hat{k}_\gamma\hat{k}_\alpha$  to eliminate the pressure  ${}^2P$ , and we also use the fluid incompressibility condition  $k_\alpha{}^2\tilde{v}_\alpha = 0$ . Third,

we ensemble average the resulting equation. As a result, we obtain

$$\begin{aligned}
\partial_t \langle {}^2\tilde{v}_{\mathbf{k}\gamma} \rangle + 3\nu k_x^2 \delta_{\gamma x}^\perp \langle {}^2\tilde{v}_{\mathbf{k}x} \rangle &= 3\nu \delta_{\gamma\alpha}^\perp i k_\beta \sum_{\substack{\mathbf{k}' \\ \mathbf{k}''=\mathbf{k}-\mathbf{k}'}} i k_x'' \left( \langle {}^1\tilde{b}_{\mathbf{k}'\alpha\beta} {}^1\tilde{v}_{\mathbf{k}''x} \rangle + \langle {}^1\tilde{b}_{\mathbf{k}'\alpha\beta} \tilde{U}_{\mathbf{k}''x} \rangle \right) \\
&+ 3\nu \delta_{\gamma x}^\perp i k_x \sum_{\substack{\mathbf{k}' \\ \mathbf{k}''=\mathbf{k}-\mathbf{k}'}} i k_\nu'' \left( \langle {}^1\tilde{b}_{\mathbf{k}'\mu\nu} {}^1\tilde{v}_{\mathbf{k}''\mu} \rangle + \langle {}^1\tilde{b}_{\mathbf{k}'\mu\nu} \tilde{U}_{\mathbf{k}''\mu} \rangle \right) \\
&- \delta_{\gamma\alpha}^\perp i k_\beta \sum_{\substack{\mathbf{k}' \\ \mathbf{k}''=\mathbf{k}-\mathbf{k}'}} \left[ \langle \tilde{U}_{\mathbf{k}'\beta} {}^1\tilde{v}_{\mathbf{k}''\alpha} \rangle + \langle \tilde{U}_{\mathbf{k}'\alpha} {}^1\tilde{v}_{\mathbf{k}''\beta} \rangle + \langle {}^1\tilde{v}_{\mathbf{k}'\alpha} {}^1\tilde{v}_{\mathbf{k}''\beta} \rangle \right]. \quad (\text{B.2})
\end{aligned}$$

Next, in order to calculate the first order quantities  ${}^1\tilde{b}_{\mathbf{k}\alpha\beta}(t)$  and  ${}^1\tilde{v}_{\mathbf{k}\alpha}(t)$ , which enter equation (B.2), we proceed as follows. First, we Fourier transform equation (3.13) in space, and we multiply the transformed equation on the left by tensor  $\delta_{\gamma\alpha}^\perp$ . We have

$$\partial_t {}^1\tilde{v}_{\mathbf{k}\gamma} + 3\nu k_x^2 \delta_{\gamma x}^\perp {}^1\tilde{v}_{\mathbf{k}x} = -3\nu k_x^2 \delta_{\gamma x}^\perp \tilde{U}_{\mathbf{k}x} + \frac{1}{5} \nu k^2 \tilde{U}_{\mathbf{k}\gamma}. \quad (\text{B.3})$$

Integrating this equation with zero initial condition,  ${}^1\mathbf{v}|_{t=0} = 0$ , we find

$${}^1\tilde{v}_{\mathbf{k}x}(t) = (\bar{\Omega} - 2\Omega) \int_0^t \tilde{U}_{\mathbf{k}x}(t') e^{-2\Omega(t-t')} dt', \quad (\text{B.4})$$

$${}^1\tilde{v}_{\mathbf{k}\gamma_{\perp x}}(t) = 2\Omega \frac{\hat{k}_x \hat{k}_{\gamma_{\perp x}}}{1 - \mu^2} \int_0^t [\tilde{U}_{\mathbf{k}x}(t') + {}^1\tilde{v}_{\mathbf{k}x}(t')] dt' + \bar{\Omega} \int_0^t \tilde{U}_{\mathbf{k}\gamma_{\perp x}}(t') dt', \quad (\text{B.5})$$

where index  $\gamma_{\perp x}$  is equal to  $y$  or  $z$ , and the viscous frequencies  $\bar{\Omega}$  and  $\Omega$  are given by equations (3.37) and (3.38). Second, we use equation (3.12) to obtain

$$\begin{aligned}
\partial_t {}^1b_{\alpha\beta} &= \partial_t ({}^1\hat{b}_\alpha {}^0\hat{b}_\beta + {}^0\hat{b}_\alpha {}^1\hat{b}_\beta) = {}^1V_{\alpha,\eta} {}^0b_{\eta\beta} + {}^1V_{\beta,\eta} {}^0b_{\eta\alpha} - 2{}^1V_{\tau,\eta} {}^0b_{\alpha\beta\tau\eta} - {}^1V_\tau {}^0b_{\alpha\beta,\tau} \\
&= (\delta_{\alpha\gamma}\delta_{\beta x} + \delta_{\beta\gamma}\delta_{\alpha x} - 2\delta_{\alpha x}\delta_{\beta x}\delta_{\gamma x}) {}^1V_{\gamma,x}. \quad (\text{B.6})
\end{aligned}$$

Now, we Fourier transform this equation and integrate the resulting equation. We have

$${}^1\tilde{b}_{\mathbf{k}\alpha\beta}(t) = ik_x(\delta_{\alpha\gamma}\delta_{\beta x} + \delta_{\beta\gamma}\delta_{\alpha x} - 2\delta_{\alpha x}\delta_{\beta x}\delta_{\gamma x}) \int_0^t [\tilde{U}_{\mathbf{k}\gamma}(t') + {}^1\tilde{v}_{\mathbf{k}\gamma}(t')] dt'. \quad (\text{B.7})$$

Finally, we substitute formulas (B.4), (B.5) and (B.7) into equation (B.2), and carry out the ensemble averagings. Because  $\tilde{\mathbf{U}}_{\mathbf{k}'}\tilde{\mathbf{U}}_{\mathbf{k}''} \propto \delta_{\mathbf{k}'',-\mathbf{k}'}$ , we find that  $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$  is equal to zero,  $\mathbf{k} = 0$ . Thus, all terms in the right-hand-side of equation (B.2) vanish. Because  $\langle {}^2\tilde{v}_{\mathbf{k}\gamma} \rangle$  is initially (at  $t = 0$ ) is zero, it stays zero in time,  $\langle {}^2\tilde{v}_{\mathbf{k}\gamma}(t) \rangle = 0$ , and using equation (3.22), we immediately obtain formula (3.52).

## Appendix C

# The Calculation of the Magnetic Energy Growth Rate, $\gamma$

It is convenient to split the right-hand-side of equation (4.17) into two terms:

$$\gamma = \gamma' + \gamma'', \quad (\text{C.1})$$

$$\gamma' = \pi \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} dk \int_{-1}^1 \mu^2 \left( 1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}} \right)^2 d\mu, \quad (\text{C.2})$$

$$\gamma'' = \pi \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} dk \int_{-1}^1 \mu^2 \left( 1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}} \right)^2 \left( 1 + \frac{2\Omega}{\Omega_{\text{rd}}} \right)^{-2} d\mu. \quad (\text{C.3})$$

Using equations (3.37), (3.38) and (3.64), we have

$$\frac{\bar{\Omega}}{\Omega_{\text{rd}}} = \frac{6k^2}{k_\nu^2}, \quad \frac{2\Omega}{\Omega_{\text{rd}}} = \frac{90k^2}{k_\nu^2} \mu^2 (1 - \mu^2). \quad (\text{C.4})$$

First, we calculate the first term,  $\gamma'$ , given by equation (C.2). Integrating over  $\mu$ , and using equation (2.11) for  $J_{0k}$ , we obtain

$$\begin{aligned}
\gamma' &= \frac{2\pi}{3} \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} \left( 1 + \frac{6k^2}{k_\nu^2} \right)^2 dk \\
&= \frac{2\pi}{3} k_0^{-3} \int_{k_0}^{k_\nu} k^4 \frac{U_0}{2k_0} (k/k_0)^{-13/3} \left( 1 + \frac{6k^2}{k_\nu^2} \right)^2 dk \\
&= \frac{\pi}{3} U_0 k_0 \left( \frac{k_\nu}{k_0} \right)^{2/3} \int_{k_0/k_\nu}^1 x^{-1/3} (1 + 6x^2)^2 dx \\
&\approx \frac{32\pi}{7} U_0 k_0 \left( \frac{k_\nu}{k_0} \right)^{2/3} = \frac{64\pi^2}{7} \left( \frac{5U_0 L}{2\pi\nu} \right)^{1/2} \frac{U_0}{L}. \tag{C.5}
\end{aligned}$$

Here, we substitute  $x = k/k_\nu$  and use the fact that the integral is dominated by the upper limit. We also use formulas  $k_0 = 2\pi/L$  ( $L$  is the system size) and  $k_\nu/k_0 = R^{3/4}$ , where  $R = U_0/k_0\nu_{\text{eff}} = 5U_0/k_0\nu$  is the Reynolds number, and  $\nu_{\text{eff}} = (1/5)\nu$  is the effective viscosity (see Section 2.2).

Second, we calculate  $\gamma''$ , given by equation (C.3), in a similar way. We have

$$\begin{aligned}
\gamma'' &= \pi \left( \frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} \left( 1 + \frac{6k^2}{k_\nu^2} \right)^2 dk \int_{-1}^1 \frac{\mu^2}{[1 + 90(k/k_\nu)^2 \mu^2 (1 - \mu^2)]^2} d\mu \\
&= 2\pi k_0^{-3} \int_{k_0}^{k_\nu} k^4 \frac{U_0}{2k_0} (k/k_0)^{-13/3} \left( 1 + \frac{6k^2}{k_\nu^2} \right)^2 dk \int_0^1 \frac{\mu^2}{[1 + 90(k/k_\nu)^2 \mu^2 (1 - \mu^2)]^2} d\mu \\
&\approx \pi U_0 k_0 \left( \frac{k_\nu}{k_0} \right)^{2/3} \int_0^1 x^{-1/3} (1 + 6x^2)^2 dx \int_0^1 \frac{\mu^2}{[1 + 90x^2 \mu^2 (1 - \mu^2)]^2} d\mu \\
&\approx 0.36\pi U_0 k_0 \left( \frac{k_\nu}{k_0} \right)^{2/3} \ll \gamma'. \tag{C.6}
\end{aligned}$$

Here, we again use equation (2.11) for  $J_{0k}$ , substitute  $x = k/k_\nu$ , and carry out the integration numerically (it clear that  $\gamma'' \ll \gamma'$  because 90 is a large number). Combining

equations (C.5), (C.6) and (C.1), we obtain equation (4.20).

Third, we calculate  $\gamma_o$ , given by equation (2.17), in order to estimate the ratio  $\gamma/\gamma_o$ . We have

$$\begin{aligned}\gamma_o &= \frac{1}{3} \left( \frac{L}{2\pi} \right)^3 \int_{-\infty}^{\infty} k^2 J_{0k} d^3 \mathbf{k} = \frac{1}{3} k_0^{-3} \int_{k_0}^{k_\nu} 4\pi k^4 \frac{U_0}{2k_0} (k/k_0)^{-13/3} dk \\ &\approx \frac{2\pi}{3} U_0 k_0 \left( \frac{k_\nu}{k_0} \right)^{2/3} \int_0^1 x^{-1/3} dx = \pi U_0 k_0 \left( \frac{k_\nu}{k_0} \right)^{2/3}.\end{aligned}\tag{C.7}$$

In the case of the kinematic dynamo, the Reynolds number is  $R = U_0/k_0\nu = U_0 L/2\pi\nu$ .

Therefore, using  $k_\nu/k_0 = R^{3/4}$ , we obtain

$$\gamma_o = \pi (2\pi)^{1/2} \left( \frac{U_0 L}{\nu} \right)^{1/2} \frac{U_0}{L}.\tag{C.8}$$

Dividing equation (4.20) by this equation, we obtain formula (4.23).

## Appendix D

### The Calculation of Coefficients

#### $\Gamma$ , $\Lambda_1$ and $\Lambda_2$ in Equation (4.88)

Before we numerically calculate  $\Gamma$ ,  $\Lambda_0$  and  $\Lambda_1$ , given by equations (4.89)–(4.93), let us first check equation (4.88). To do this, we use equation (2.20) and equation (4.88) to obtain the change of the total magnetic energy  $\mathcal{E}$  in time,

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} &= \frac{1}{2} \int_0^\infty \frac{\partial M}{\partial t} dk = \frac{1}{2} \frac{\Gamma}{5} \int_0^\infty \left[ k^2 \frac{\partial^2 M}{\partial k^2} - (\Lambda_1 - 1)k \frac{\partial M}{\partial k} + \Lambda_0 M \right] dk \\ &= \frac{1}{2} \frac{\Gamma}{5} \left[ 2 + (\Lambda_1 - 1) + \Lambda_0 \right] \int_0^\infty M dk = \frac{\Gamma}{5} (\Lambda_1 + \Lambda_0 + 1) \mathcal{E} \\ &= 2\gamma \mathcal{E},\end{aligned}\tag{D.1}$$

where the growth rate  $\gamma$  is given by equation (4.17). To obtain the second line of this equation, we integrate the first line by parts. To obtain the last (third) line of this equation, we use equations (4.90) and (4.91). Equation (D.1) coincides with equation (4.16), as one might expect.

Now, let us refer to equations (4.89)–(4.93). Using equation (2.11) for  $J_{0k}$ , and

changing the integration variables,  $k \rightarrow x = k/k_\nu$ , we obtain

$$\frac{\bar{\Omega}}{\Omega_{\text{rd}}} = 6x^2 \quad (\text{D.2})$$

$$\frac{2\Omega}{\Omega_{\text{rd}}} = 90x^2 \sin^2 \theta \cos^2 \varphi (1 - \sin^2 \theta \cos^2 \varphi), \quad (\text{D.3})$$

$$\Gamma = \left( \frac{U_0 L}{\nu} \right)^{1/2} \frac{U_0}{L} \Gamma', \quad (\text{D.4})$$

$$\begin{aligned} \Gamma' = & \frac{5}{2} \left( \frac{5\pi}{2} \right)^{1/2} \int_0^1 x^{-1/3} dx \int_0^\pi d\theta \sin^3 \theta \cos^2 \theta \\ & \times \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}} \right)^2 \left\{ 1 - \left[ 1 - \left( 1 + \frac{2\Omega}{\Omega_{\text{rd}}} \right)^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}, \end{aligned} \quad (\text{D.5})$$

$$\begin{aligned} \Lambda_1 = & -3 + \frac{5}{\Gamma'} \left( \frac{5\pi}{2} \right)^{1/2} \int_0^1 x^{-1/3} dx \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}} \right)^2 \\ & \times \left\{ 2 \cos^2 \theta \cos^2 \varphi + \frac{1}{2} \sin^2 \theta - \left[ 1 - \left( 1 + \frac{2\Omega}{\Omega_{\text{rd}}} \right)^{-2} \right] \right. \\ & \times \left. \left( 2 + \frac{1}{2} \frac{\sin^2 \theta}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right) \cos^2 \theta \cos^2 \varphi \right\}, \end{aligned} \quad (\text{D.6})$$

$$\begin{aligned} \Lambda_0 = & 2 + \frac{5}{\Gamma'} \left( \frac{5\pi}{2} \right)^{1/2} \int_0^1 x^{-1/3} dx \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi \left( 1 + \frac{\bar{\Omega}}{\Omega_{\text{rd}}} \right)^2 \\ & \times \left\{ 2 \sin^2 \theta \cos^2 \varphi - \frac{1}{2} \sin^2 \theta + \left[ 1 - \left( 1 + \frac{2\Omega}{\Omega_{\text{rd}}} \right)^{-2} \right] \right. \\ & \times \left. \left( \cos^2 \theta - \sin^2 \theta + \frac{1}{2} \frac{\sin^2 \theta \cos^2 \theta}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right) \cos^2 \varphi \right\}. \end{aligned} \quad (\text{D.7})$$

Here we use formulas  $k_0 = 2\pi/L$  ( $L$  is the system size) and  $k_\nu/k_0 = R^{3/4}$ , where  $R = U_0/k_0\nu_{\text{eff}} = 5U_0/k_0\nu$  is the Reynolds number, and  $\nu_{\text{eff}} = (1/5)\nu$  is the effective viscosity (see Section 2.2). We calculate the triple integrals in equations (D.5)–(D.7) numerically. We obtain

$$\Gamma' \approx 104, \quad \Lambda_1 \approx 2.13, \quad \Lambda_0 \approx 5.21, \quad (\text{D.8})$$

which, immediately give us equations (4.94)–(4.96).

It is interesting that if  $\Omega_{\text{rd}} = \Omega_{\text{rd}}(k)$  [it depends only on  $k$ , not on  $\mu^2$ ], then in the limit  $\Omega_{\text{rd}}/\nu_{\text{eff}}k_\nu^2 \rightarrow 0$  (weak rotational damping) we have  $\Lambda_1 = 2$  and  $\Lambda_0 = 5$ . The results given by equation (D.8) are close to these results.

## Appendix E

# The Derivation of the Green's Function Solution (4.99)

To solve equation (4.88), we Laplace transform it in time,  $t \rightarrow s$  and  $M(t, k) \rightarrow \tilde{M}(s, k)$ . We assume that  $M(t, k)$  is zero at  $t = 0$  and that we know  $M(t, k_{\text{ref}})$  at some reference wave number  $k = k_{\text{ref}}$  at  $t > 0$ . To simplify our notations we set  $k_{\text{ref}} = 1$ , so that  $M(t, k_{\text{ref}}) = M(t, 1)$ . As a result, we have

$$\frac{5}{\Gamma} s \tilde{M} = k^2 \frac{\partial^2 \tilde{M}}{\partial k^2} - (\Lambda_1 - 1) k \frac{\partial \tilde{M}}{\partial k} + \Lambda_0 \tilde{M}. \quad (\text{E.1})$$

We consider the following solution of this equation:

$$\tilde{M}(s, k) = \tilde{M}(s, 1) k^{(\Lambda_1/2) \pm \sqrt{5s/\Gamma - (\Lambda_0 - \Lambda_1^2/4)}}, \quad (\text{E.2})$$

where

$$\tilde{M}(s, 1) = \int_0^\infty e^{-st'} M(t', 1) dt' \quad (\text{E.3})$$

is the Laplace coefficient of function  $M(t, 1)$ . This solution satisfies  $\tilde{M}|_{k=1} = \tilde{M}(s, 1)$ . Now, the minus sign in the exponent in equation (E.2) should be chosen to satisfy the boundary condition at infinity in  $k$ ,  $\tilde{M}(s, \infty) = 0$ , [27]. The Laplace inversion of  $\tilde{M}(s, k)$ , given by equations (E.2) and (E.3), is

$$\begin{aligned} M(t, k) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \tilde{M}(s, k) ds \\ &= \frac{1}{2\pi i} \int_0^\infty dt' \int_{c-i\infty}^{c+i\infty} M(t', 1) e^{s(t-t')} k^{\Lambda_1/2} k^{-\sqrt{5s/\Gamma - (\Lambda_0 - \Lambda_1^2/4)}} ds. \end{aligned} \quad (\text{E.4})$$

Changing the integration variables

$$t' \rightarrow \tau = t - t', \quad s \rightarrow z = \sqrt{5s/\Gamma - (\Lambda_0 - \Lambda_1^2/4)}, \quad (\text{E.5})$$

we obtain

$$M(t, k) = \frac{\Gamma}{5\pi i} k^{\Lambda_1/2} \int_{-\infty}^t M(t - \tau, 1) e^{(\Gamma/5)(\Lambda_0 - \Lambda_1^2/4)\tau} d\tau \int_{-i\infty}^{i\infty} z e^{(\Gamma\tau/5)z^2 - z \ln k} dz. \quad (\text{E.6})$$

Here, the  $z$  integration can be taken as the imaginary vertical axis in the complex  $z$ -plane. Carrying out the  $z$  integration, we obtain

$$M(t, k) = \left(\frac{5}{4\pi}\right)^{1/2} \frac{k^{\Lambda_1/2} \ln k}{\Gamma^{1/2}} \int_{-\infty}^t M(t - \tau, 1) \frac{e^{(\Gamma/5)(\Lambda_0 - \Lambda_1^2/4)\tau - 5 \ln^2 k / 4\Gamma\tau}}{\tau^{3/2}} d\tau. \quad (\text{E.7})$$

This function satisfies equation (4.88), but does not satisfy the initial condition equation,  $M(0, k) = 0$ . It is clear that function

$$M(t, k) = \left(\frac{5}{4\pi}\right)^{1/2} \frac{k^{\Lambda_1/2} \ln k}{\Gamma^{1/2}} \int_0^t M(t - \tau, 1) \frac{e^{(\Gamma/5)(\Lambda_0 - \Lambda_1^2/4)\tau - 5 \ln^2 k / 4\Gamma\tau}}{\tau^{3/2}} d\tau \quad (\text{E.8})$$

satisfies both equations. Because  $M(t - \tau, 1)$  is zero if  $\tau > t$ , we can extend the upper integration limit in equation (E.8) to infinity. Therefore, this equation is equivalent to equations (4.98) and (4.99), after we replace  $k$  by  $k/k_{\text{ref}}$  and  $t - \tau$  by  $t'$ .

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